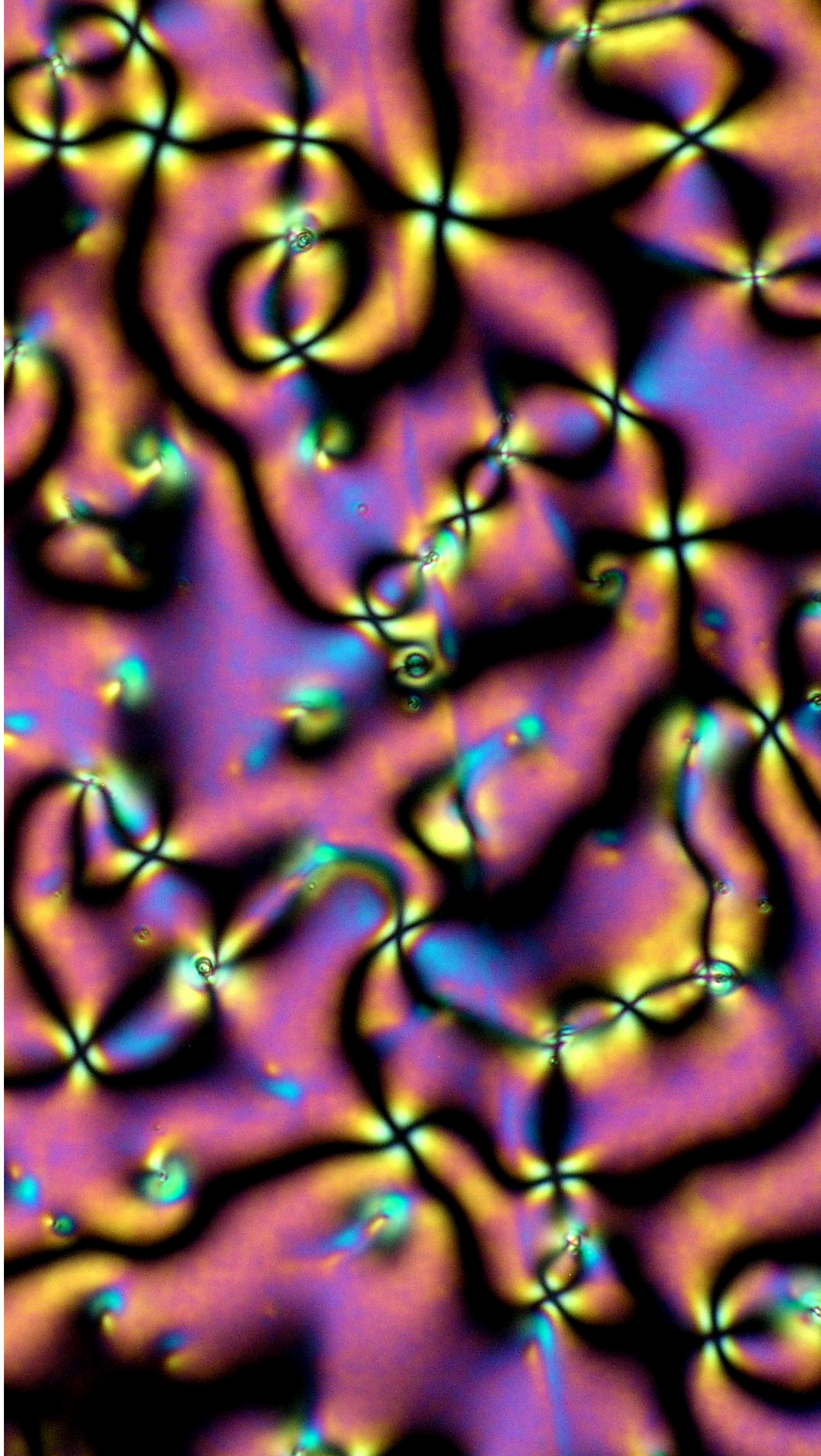
GENERALIZED SYMMETRIES AND ORDERED PHASES

Sal Pace (MIT)

Based on:

- **SP**, arXiv: 2308.05730
- **SP**, C Zhu, A Beaudry, and X-G Wen, *in preparation*



A SYMMETRY RENAISSANCE

Our understanding of symmetry has been revolutionized through modern generalizations

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Our understanding of symmetry has been revolutionized through modern generalizations

Higher-form symmetry

Non-invertible symmetry

Dipole symmetry

Loop group symmetry

Subsystem symmetry

Higher-group symmetry

Biform symmetry

Harmonic symmetry

Framed symmetry

Modulated symmetry

A SYMMETRY RENAISSANCE

Our understanding of symmetry has been revolutionized through modern generalizations

Framed symmetry

Modulated symmetry

MODERN VIEW ON SYMMETRIES

Topological defect = symmetry defect

Passes the duck test!

If it looks like a duck, swims like a duck, and quacks like a duck, then it probably is a duck.

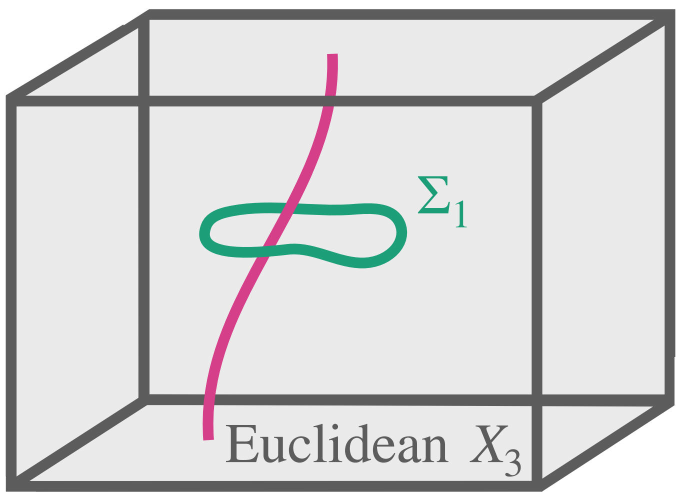
- There's a symmetry operator that commutes with H
- Objects carrying the symmetry charge can condense, causing spontaneous symmetry breaking
- Can have 't Hooft anomalies

SUPERFLUIDS AT LOW ENERGY

Superfluid in $D = 3$ Euclidean spacetime X_3

$$U(1) \xrightarrow{\text{ssb}} 1 \qquad\qquad \theta : X_3 \to \mathscr{M} = \mathbb{R}/\mathbb{Z}$$

➤ 1-dimensional vortices (defects) detected by $\Sigma_1 \subset X_3$:



$$Q(\Sigma_1) = \int_{\Sigma_1} \mathrm{d}\theta \in \pi_1(\mathscr{M}) \simeq \mathbb{Z}$$

Vortex is a singularity in the order parameter field $\theta(x)$ and is not dynamical at low energy

SUPERFLUIDS AT LOW ENERGY

At low energies, there's a $U(1)$ 1-form symmetry generated by the topological defect

$$T^{(\alpha)}(\Sigma_1) = \exp(\mathrm{i}\alpha Q(\Sigma_1)) \qquad\qquad Q(\Sigma_1) = \int_{\Sigma_1} \mathrm{d}\theta \in \mathbb{Z}$$

➤ Vortex is charged under the $U(1)^{(1)}$ symmetry

In Lorentzian X_3 the vortex defect in space is an operator creating a loop in space (*membrane in spacetime*)

➤ $U(1)^{(1)}$ symmetry ensures "loop number" is conserved

IN THIS TALK

Explore emergent generalized symmetry $\mathcal{S}_\pi$ in *generic* ordered phases and its spontaneous breaking

Why should you care?

- cond-mat: ordered phases are common and $\mathcal{S}_\pi$ can predict exotic disordered phases
- hep-th: $\mathcal{S}_\pi$ describes the topological sectors of NLσMs
- math-ph: $\mathcal{S}_\pi$ is related to higher representations of a higher group and the twisted fibrations in a Postnikov tower

IN THIS TALK

Explore emergent generalized symmetry $\mathcal{S}_\pi$ in *generic* ordered phases and its spontaneous breaking

1. General features of ordered phases and homotopy defects
2. Emergence of generalized symmetries and their symmetry categories
3. Spontaneous symmetry breaking and nontrivial disordered phases

ORDERED PHASES

A phase where an ordinary internal symmetry G is spontaneously broken

- Universal features determined by the SSB pattern

$$G \xrightarrow{\text{ssb}} H \subset G$$

- Ground states labeled by order parameter manifold

$$\mathscr{M} = G/H = \{gH : g \in G\}$$

- There can be gapped objects called Homotopy defects, characterized by the topology of $\mathscr{M}$

  (e.g., domain walls, vortices, hedgehogs, etc)

HOMOTOPY DEFECTS IN THE IR

Continuous G: Effective field theory is a nonlinear σ model with target space $\mathscr{M} = G/H$ describing the Goldstone modes

[*Callan, Coleman, Wess, Zumino (1969)*
Watanabe, Murayama (2014)]

- Homotopy defects are singularities of the order parameter field $U : X \to \mathscr{M}$.

Finite G: Effective field theory is a TQFT describing the SSB ground states

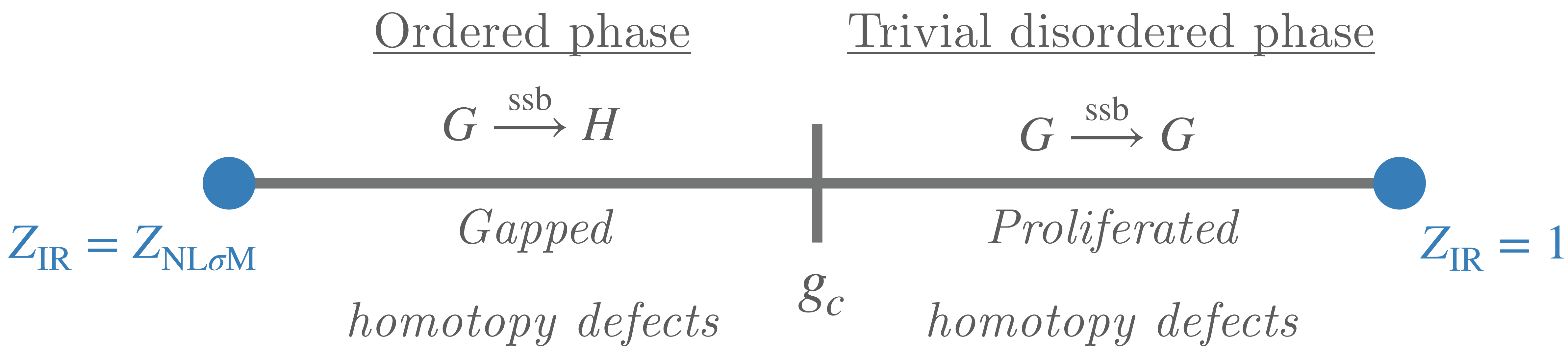
- Homotopy defects are certain G symmetry defects of the TQFT

Homotopy defects are not dynamical

HOMOTOPY DEFECTS IN THE UV

In a generic UV theory (e.g., lattice models), Homotopy defects are dynamical

- The prototypical phase diagram:



- Proliferating homotopy defects drives phase transitions.
  (*From an IR perspective, proliferation is like summing over all defect insertions*)

HOMOTOPY DEFECT CLASSIFICATION

Homotopy defects detected by the k-submanifold Σ_k are classified by

$$\mathrm{Maps}(\Sigma_k, \mathscr{M} = G/H)/\mathrm{homotopy}$$

- $\Sigma_k \simeq S^k$: defects are detected via linking, are codimension $k+1$, and classification is based on homotopy groups

$$\pi_k(\mathscr{M})/\alpha_k, \qquad k = 1,\, 2, \cdots,\, D-2,\, D-1$$

where $\alpha_k : \pi_1(\mathscr{M}) \to \mathrm{Aut}(\pi_k(\mathscr{M}))$ [*Mermin (1979)*]

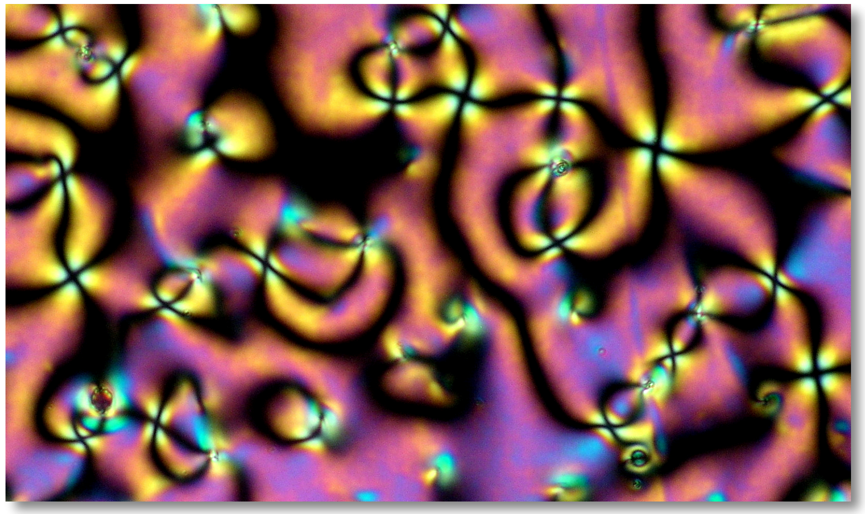
- e.g., α_1 is the inner automorphism, so codimension 2 homotopy defects classified by conjugacy classes $\mathrm{Cl}(\pi_1(\mathscr{M}))$

NEMATIC LIQUID CRYSTAL

Ordered phase with SSB pattern

$$SO(3) \xrightarrow{\text{ssb}} O(2)$$

$$\mathscr{M} = SO(3)/O(2) \simeq \mathbb{RP}^2$$



➤ In $D = 3$ dimensional spacetime: [*Volovik, Mineev (1977)*]

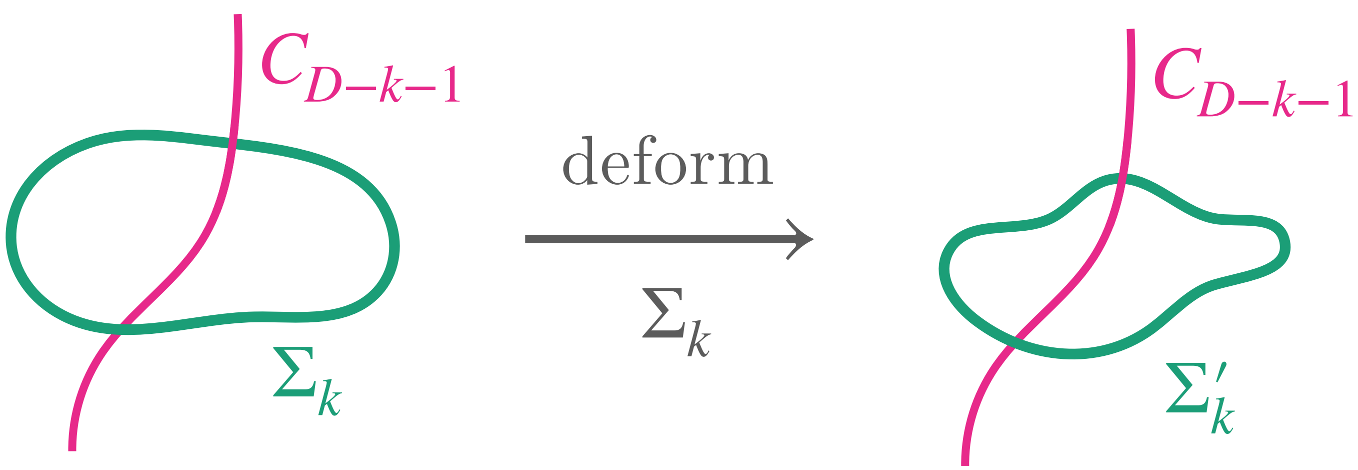
$\pi_0(\mathbb{RP}^2) = 0$ $\pi_1(\mathbb{RP}^2) = \mathbb{Z}_2$ $\pi_2(\mathbb{RP}^2) = \mathbb{Z}$

$\alpha_2 : \pi_1(\mathbb{RP}^2) \to \text{Aut}(\pi_2(\mathbb{RP}^2))$ flips sign of $\pi_2(\mathbb{RP}^2)$

$\mathbb{Z}_2$ strings and $\mathbb{Z}_{\geq 0}$ particles

TOPOLOGICAL DEFECTS

Since homotopy defects are classified by $\mathrm{Maps}(\Sigma_k, \mathscr{M})$ modulo homotopy, the defects detecting homotopy defects have topological properties.



$\mathrm{Maps}(\Sigma_k, \mathscr{M})/\mathrm{homotopy}$ depend only on $\mathrm{Link}(\Sigma_k, C_{D-k-1})$ when $\partial C_{D-k-1} = \varnothing$

When homotopy defects cannot be cut open (cannot end [Hsin (2022)])

They are detected by topological defects $\Longrightarrow$ They carry symmetry charge of a symmetry $\mathcal{S}_\pi$

TOPOLOGICAL CURRENTS

For abelian homotopy defects classified by $\pi_k(\mathscr{M} = G/H) = \mathbb{Z}$, number of homotopy defects detected by $\Sigma_k \subset X$ [*D'Hoker, Weinberg (1994)*]

$$Q(\Sigma_k) = \int_{\Sigma_k} \Omega^{(k)} \in \mathbb{Z}$$

- $\Omega^{(k)}$ is generator of $H^k_{\mathrm{dR}}(\mathscr{M})$ pulled back to X (i.e., $\Omega^{(1)} = \mathrm{d}\theta$)
- $\mathrm{d}\Omega^{(k)} = \hat{C}$, the Poincaré dual of the homotopy defect's location
- $\mathcal{S}_\pi$: topological defects generating $U(1)^{(D-k-1)}$ symmetries:

$$T^{(\alpha)}(\Sigma_k) = \exp(\mathrm{i}\alpha Q(\Sigma_k))$$

Gaiotto, Kapustin, Seiberg Willet (2015)
Grozdanov, Poovuttikul (2018)
Delacrétaz, Hofman, Mathys (2020)
Armas, Jain (2020)
Brauner (2021)

EMERGENT SYMMETRIES

In generic UV models, homotopy defects are dynamical

- High energy processes cut open homotopy defects
- $\mathcal{S}_\pi$ is not a symmetry in the UV

In the IR, homotopy defects are not dynamical

- $\mathcal{S}_\pi$ is a symmetry in the IR

Generic ordered phases have an emergent $\mathcal{S}_\pi$ symmetry

- We will always implicitly refer to $\mathcal{S}_\pi$ at the lowest energy scale (the deep IR)

GENERALIZED SYMMETRIES IN PRACTICE

<u>UV Scale</u> → <u>IR Scale</u>

$\mathscr{S}_{\text{UV}}$ $\mathscr{S}_{\text{mid–IR}}$ $\mathscr{S}_{\text{IR}}$

$\mathscr{S}_{\text{UV}}$ includes ordinary/no symmetries, but $\mathscr{S}_{\text{mid–IR}}$ and $\mathscr{S}_{\text{IR}}$ can include emergent generalized symmetries

- Emergent higher-form symmetries are exact symmetries, not approximate symmetries *Iqbal, McGreevy (2022)* ***SP**, Wen (2023)* *McGreevy (2022)* *Cherman, Jacobson (2023)* *Cheng, Seiberg (2023)*
- The generalized Landau paradigm is really a classification scheme about emergent generalized symmetries

THE SYMMETRY $\mathcal{S}_\pi$

What is this generalized symmetry?

- $\mathcal{S}_\pi$ = magnetic symmetry of $\mathbb{G}_\pi^{(D-1)}$ higher gauge theory
- Finite homotopy defect types: $\mathcal{S}_\pi = (D-1)\text{-Rep}(\mathbb{G}_\pi^{(D-1)})$

Examples with $G = SO(3)$:

D	SSB Pattern	$\mathcal{S}_\pi$
3	$SO(3) \xrightarrow{\text{ssb}} 1$	$\mathbb{Z}_2^{(1)} \quad (\mathcal{S}_\pi = 2\text{-Rep}(\mathbb{Z}_2))$
3	$SO(3) \xrightarrow{\text{ssb}} \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathcal{S}_\pi = 2\text{-Rep}(Q_8)$
4	$SO(3) \xrightarrow{\text{ssb}} SO(2)$	$0 \to \mathbb{Z}^{(2)} \to \mathbb{G}_\pi^{(3)} \to \mathbb{Z}^{(1)} \to 0$

ABELIAN HOMOTOPY DEFECTS

Consider general abelian homotopy defects

- Defects of different dimension are independent from one another
- Trivial $\alpha_k : \pi_1(\mathscr{M}) \to \mathrm{Aut}(\pi_k(\mathscr{M})) \implies$ classified by $\pi_k(\mathscr{M})$.

Each $\pi_k(\mathscr{M})$ describes symmetry charges of a $(D-k-1)$-form symmetry.

- $(D-k-1)$-form symmetry group is the Pontryagin dual of $\pi_k(\mathscr{M})$

$$G^{(D-k-1)} = \mathrm{Hom}(\pi_k(\mathscr{M}), U(1))$$

CODIMENSION 2 HOMOTOPY DEFECTS

Since we only care about $\pi_1(\mathscr{M})$, let's truncate $\mathscr{M}$ to $\mathscr{M}_{\tau\leq 1}$:

$$\pi_k(\mathscr{M}_{\tau\leq 1}) = \begin{cases} \pi_k(\mathscr{M}) & k = 1 \\ 0 & \text{else} \end{cases}$$

$$\mathscr{M}_{\tau\leq 1} = B\pi_1(\mathscr{M})$$

$\mathcal{S}_\pi$ from codimension 2 homotopy defects of $\mathscr{M}$ is the same as $\mathcal{S}_\pi$ from $\mathscr{M}_{\tau\leq 1}$

- These homotopy defects are $\pi_1(\mathscr{M})$ magnetic defects
- $\mathcal{S}_\pi =$ the magnetic symmetry of $\pi_1(\mathscr{M})$ gauge theory
- Finite $\pi_1(\mathscr{M})$: $\mathcal{S}_\pi = (D-1)\text{-Rep}(\pi_1(\mathscr{M}))$

GENERAL CONNECTED $\mathscr{M}$

Since $\pi_k(\mathscr{M})$ homotopy defects for $k > D - 1$ are absent in D dimensions, we truncate $\mathscr{M}$ to $\mathscr{M}_{\tau \le D-1}$:

$$\pi_k(\mathscr{M}_{\tau \le n}) = \begin{cases} \pi_k(\mathscr{M}) & 1 \le k \le n \\ 0 & \text{else} \end{cases}$$

$\mathcal{S}_\pi$ from $\mathscr{M}$ is the same as $\mathcal{S}_\pi$ from $\mathscr{M}_{\tau \le D-1}$

- $\mathscr{M}_{\tau \le n}$ is called the nth Postnikov stage of $\mathscr{M}$
- Model connected homotopy n-types.

GENERAL CONNECTED $\mathscr{M}$

Postnikov stages obey the fibrations $(2 \leq n \leq D-1)$

$$B^n\pi_n(\mathscr{M}) \to \mathscr{M}_{\tau\leq n} \to \mathscr{M}_{\tau\leq n-1}$$

Classified by the twisted $(n+1)$-cocycle [*Baez, Shulman (2009)*]

$$[\beta^{n+1}] \in H^{n+1}_{\alpha_n}(\mathscr{M}_{\tau\leq n-1}, \pi_n(\mathscr{M})) \qquad \alpha_n : \pi_1(\mathscr{M}) \to \mathrm{Aut}(\pi_n(\mathscr{M}))$$

- $\mathscr{M}_{\tau\leq n}$ is the classifying space of an n-group: $\mathscr{M}_{\tau\leq n} = B\mathbb{G}^{(n)}_\pi$,

$$\mathbb{G}^{(n)}_\pi = (\pi_1(\mathscr{M})\ ;\ \pi_2(\mathscr{M}), \alpha_2, \beta^3\ ;\ \cdots\ ;\ \pi_n(\mathscr{M}), \alpha_n, \beta^{n+1})$$

- Homotopy defects are $\mathbb{G}^{(D-1)}_\pi$ magnetic defects
- $\mathscr{S}_\pi =$ magnetic symmetry of $\mathbb{G}^{(D-1)}_\pi$ higher gauge theory
- For finite $\mathbb{G}^{(D-1)}_\pi$, it is $\mathscr{S}_\pi = (D-1)\text{-Rep}(\mathbb{G}^{(D-1)}_\pi)$

CHECK IN

In the deep IR, homotopy defects carry symmetry charge of an emergent generalized symmetry $\mathcal{S}_\pi$

- $\mathcal{S}_\pi$ = magnetic symmetry of $\mathbb{G}_\pi^{(D-1)}$ higher gauge theory
- $\mathcal{S}_\pi$ includes invertible and non-invertible, 0-form and higher-form symmetries.

What are some physical applications of $\mathcal{S}_\pi$?

1. Spontaneous symmetry breaking *(we'll discuss here)*
2. Mixed 't Hooft anomaly with G *(we won't discuss here)*

SPONTANEOUSLY BREAKING $\mathcal{S}_\pi$

$\mathcal{S}_\pi$ is not spontaneously broken in the ordered phase

- If it were, ordered phases would have GSD dependent on space's topology and exotic gapless modes
- Homotopy defects are gapped extended objects in the spectrum and are confined in the IR (*i.e., area law*)

$\mathcal{S}_\pi$ can spontaneously break, driving a transition out of the ordered phase

- Typically leads to an exotic phase of matter
- Homotopy defects will deconfine (*i.e., perimeter law*)

SPONTANEOUSLY BREAKING $\mathcal{S}_\pi$

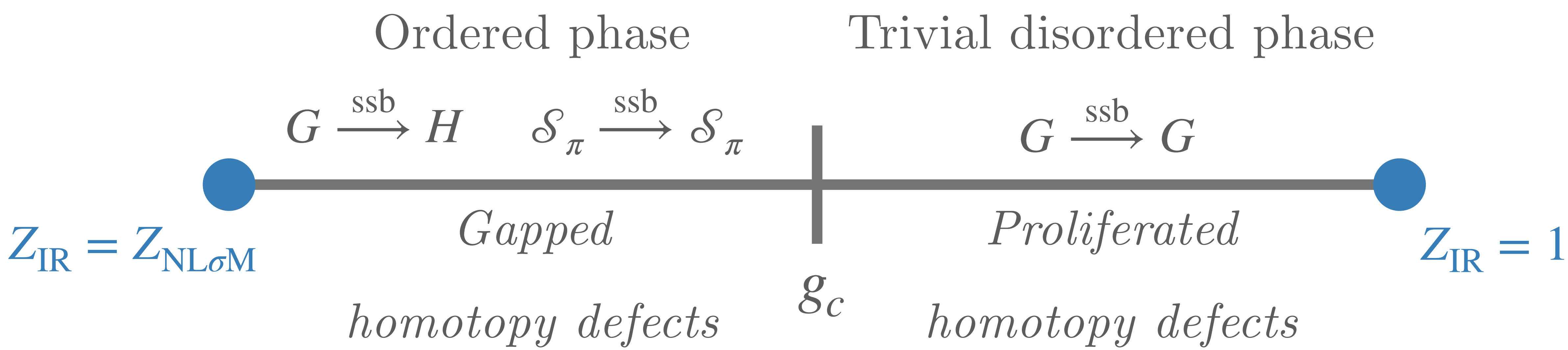
What happens to the microscopic G symmetry when spontaneously breaking $\mathcal{S}_\pi$?

<u>A physical argument</u>

- Ordered vacua ($G \xrightarrow{\text{ssb}} H$) want to confine homotopy defects
- $\mathcal{S}_\pi$ SSB vacua have a $\mathcal{S}_\pi$ symmetry charge condensate that wants to deconfine homotopy defects
- The latter contradicts the former, so spontaneously breaking $\mathcal{S}_\pi$ must restore G

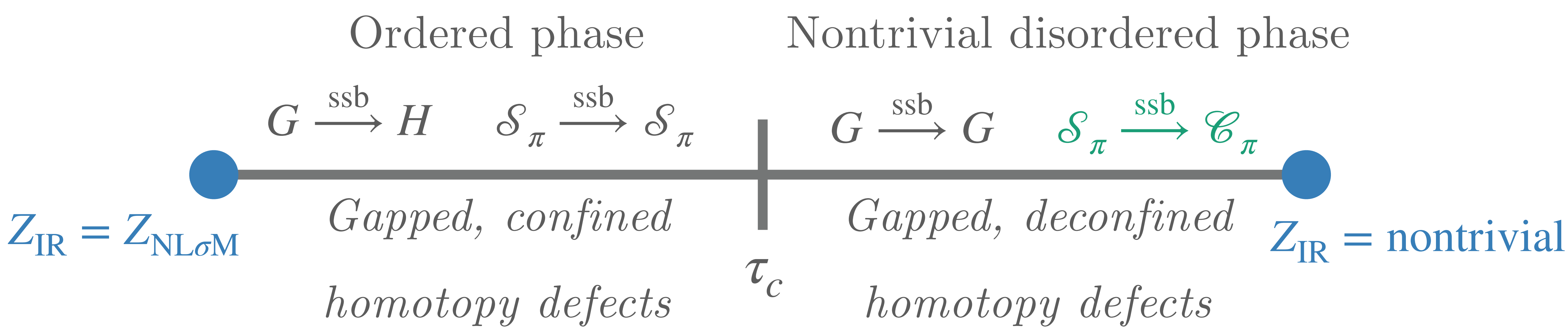
TWO TYPES OF DISORDERED PHASES

Without defect suppression:



With perfect defect suppression:

Chubukov, Senthil, Sachdev (1994)
Lammert, Rokhsar, Toner (1995)
Motrunich, Vishwanath (2004)
Grover, Senthil (2011)
Xu, Ludwig (2012)
Zhu, Lan, Wen (2019)



THE POWER OF SYMMETRY

$\mathcal{S}_\pi$ is a non-perturbative tool to identify exotic phases neighboring ordered phases

D	Ordered phase $G \xrightarrow{\text{ssb}} H$	Nontrivial disordered phase $\mathcal{S}_\pi \xrightarrow{\text{ssb}} \mathscr{C}_\pi$
4	$U(1) \xrightarrow{\text{ssb}} 1$	none
4	$U(1) \times U(1) \xrightarrow{\text{ssb}} 1$	$U(1)^{(1)} \xrightarrow{\text{ssb}} 1$
3	$SO(3) \xrightarrow{\text{ssb}} 1$	$\mathbb{Z}_2^{(1)} \xrightarrow{\text{ssb}} 1$
3	$SO(3) \xrightarrow{\text{ssb}} \mathbb{Z}_2 \times \mathbb{Z}_2$	$\text{Rep}(Q_8)^{(1)} \xrightarrow{\text{ssb}} 1$

- For finite $\mathbb{G}_\pi^{(D-1)}$: Nontrivial disordered phase is the deconfined phase of untwisted $\mathbb{G}_\pi^{(D-1)}$ higher gauge theory

FUN WITH $G = SO(3)$: PART I

Consider SSB pattern $SO(3) \xrightarrow{\text{ssb}} H$ with finite H in $D = 3$

$$\mathscr{M} = SO(3)/H$$

$$\pi_0(\mathscr{M}) = 0 \qquad \pi_1(\mathscr{M}) = \tilde{H} \qquad \pi_2(\mathscr{M}) = 0$$

where $\tilde{H}$ is the cover of H that lifts it to a subgroup of $SU(2)$.

- e.g., $H = \mathbb{Z}_N = \tilde{H}$ and $H = \mathbb{Z}_2 \times \mathbb{Z}_2 \implies \tilde{H} = Q_8$
- $1D$ homotopy defects classified by conjugacy classes $\mathrm{Cl}(\tilde{H})$
- Emergent symmetry $\mathcal{S}_\pi = 2\text{-Rep}(\tilde{H})$

Let's build a Euclidean lattice model with the $\mathcal{S}_\pi$ SSB phase

FUN WITH $G = SO(3)$: PART I

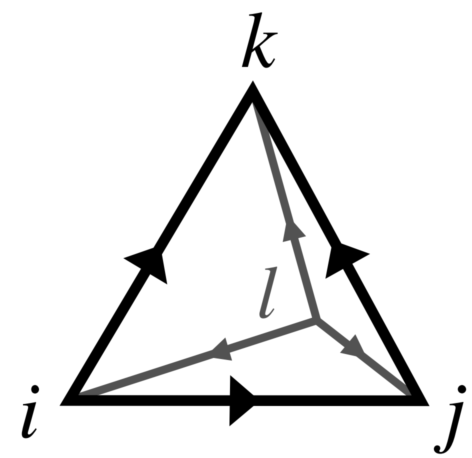
Consider the order parameter presentation

- On lattice sites i, $\tilde{G} = SU(2)$ rotors $z_i \in \mathbb{C}^2$ with $z_i^\dagger z_i = 1$. $SO(3)$ realized as $SU(2)$ transforming z_i in $\square$ of $SU(2)$
- On lattice edges (ij), $\tilde{H}$ gauge fields a_{ij} in $\square$ of $SU(2)$ restricted to $\tilde{H}$.

Why?

- Gauge redundancy $z_i \sim \tilde{h}_i z_i$, $a_{ij} \sim \tilde{h}_i a_{ij} \tilde{h}_j^{-1}$ restricts physical z_i values in $SU(2)/\tilde{H} = SO(3)/H \equiv \mathcal{M}$
- 1D homotopy defects realized as $\tilde{H}$ gauge fluxes

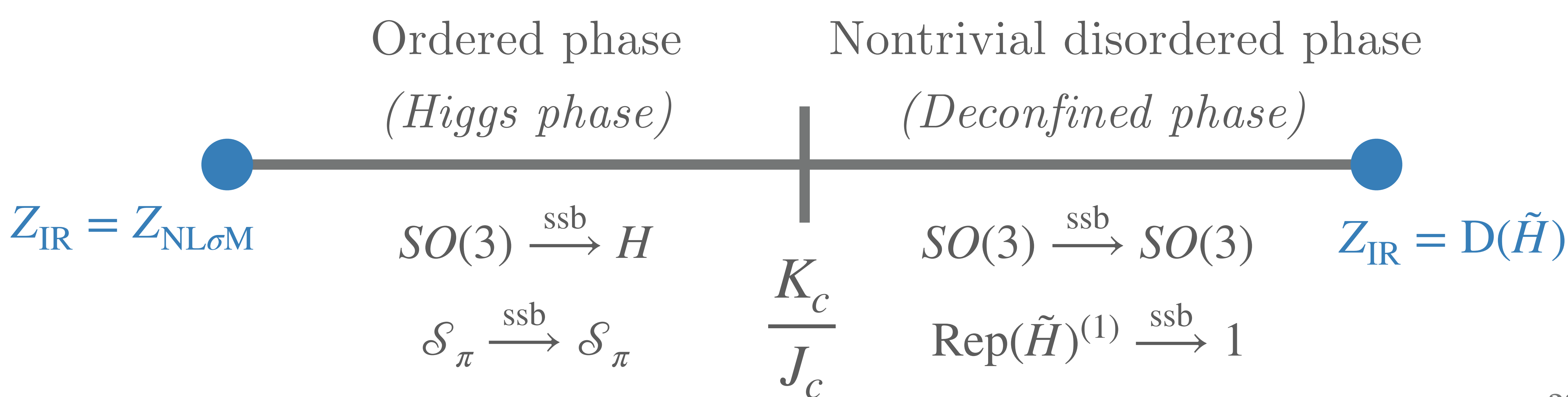
FUN WITH $G = SO(3)$: PART I



$$S = -J\sum_{(ij)} z_i^\dagger a_{ij} z_j + K \sum_{(ijk)} \mathrm{Tr}\left[(\delta a)_{ijk}\right]$$

$$(\delta a)_{ijk} = a_{ij} a_{jk} a_{ik}^{-1}$$

- J term wants z_i (gauge charges) to condense
- K term penalizes $\pi_1(\mathscr{M})$ homotopy defects ($\tilde{H}$ gauge fluxes)



FUN WITH $G = SO(3)$: PART II

Consider SSB pattern $SO(3) \xrightarrow{\text{ssb}} SO(2)$ $[\mathscr{M} = S^2]$ in $D = 4$

$\pi_0(S^2) = 0$ $\qquad$ $\pi_1(S^2) = 0$ $\qquad$ $\pi_2(S^2) = \mathbb{Z}$ $\qquad$ $\pi_3(S^2) = \mathbb{Z}$

- Because $\pi_1(S^2)$ is trivial

$$\mathbb{G}^{(3)}_{\pi} = (\pi_2(S^2)\ ;\ \pi_3(S^2),\ \beta^4) \qquad [\beta^4] \in H^4(B^2\mathbb{Z}, \mathbb{Z})$$

- Consider 2 & 3 cochains $x^{(2)}$ & $x^{(3)}$ on $B\mathbb{G}^{(3)}_{\pi} = S^2_{\tau \leq 3}$

$$\mathrm{d}x^{(2)} = 0 \qquad \mathrm{d}x^{(3)} = x^{(2)} \cup x^{(2)} \equiv \beta^4(x^{(2)})$$

$\mathcal{S}_{\pi}$ = magnetic symmetry of $\mathbb{G}^{(3)}_{\pi}$ gauge theory

- Non-invertible symmetry since $\mathbb{G}^{(3)}_{\pi}$ does not have a Pontryagin dual 3-group [*Chen, Tanizaki (2023)*]

FUN WITH $G = SO(3)$: PART II

To construct effective theory for the nontrivial disordered phase, consider the $\mathbb{CP}^1$ presentation of the S^2 NLσM

- $U(1)$ 1-form gauge field $A^{(1)}$

$$\int_{S^2} \frac{1}{2\pi} \mathrm{d}A^{(1)} \in \pi_2(S^2) \qquad \int_{S^3} \frac{1}{4\pi^2} A^{(1)} \wedge \mathrm{d}A^{(1)} \in \pi_3(S^2)$$

Motivates us to introduce the $U(1)$ 2-form gauge field $B^{(2)}$ and gauge invariant field strengths

$$F^{(2)} = \mathrm{d}A^{(1)} \qquad H^{(3)} = \frac{1}{2\pi} A^{(1)} \wedge \mathrm{d}A^{(1)} + \mathrm{d}B^{(2)}$$

$$A^{(1)} \sim A^{(1)} + \mathrm{d}f_1^{(0)} \qquad B^{(2)} \sim B^{(2)} + \mathrm{d}f_2^{(1)} - \frac{1}{2\pi} f_1^{(0)} \mathrm{d}A^{(1)}$$

FUN WITH $G = SO(3)$: PART II

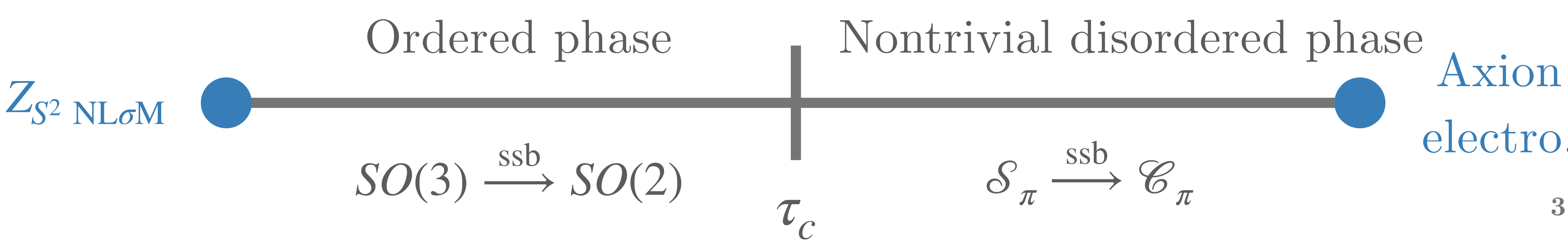
Effective field theory of the nontrivial disordered phase

$$S_{\text{IR}} = \int_{M_4} \frac{1}{2e^2} F^{(2)} \wedge \star F^{(2)} + \frac{1}{4\pi v^2} H^{(3)} \wedge \star H^{(3)}$$

Dualizing $B^{(2)}$ to the compact boson $\phi^{(0)}$

$$S_{\text{IR}} = \int_{M_4} \frac{1}{2e^2} F^{(2)} \wedge \star F^{(2)} + \frac{v^2}{2} \mathrm{d}\phi^{(0)} \wedge \star \mathrm{d}\phi^{(0)} + \frac{1}{4\pi^2} \phi^{(0)} F^{(2)} \wedge F^{(2)}$$

➤ Massless axion electrodynamics enriched by $SO(3)$!



SUMMARY

SP, arXiv: 2308.05730

SP, C Zhu, A Beaudry, and X-G Wen, *in preparation*

Generalized symmetries emerge in ordered phases

- Symmetry charge carried by homotopy defects
- Symmetry defects described by $(D-1)$-representations of $\mathbb{G}_{\pi}^{(D-1)} = (\pi_1(\mathscr{M})\ ;\ \pi_2(\mathscr{M}), \alpha_2, \beta^3\ ;\ \cdots\ ;\ \pi_{D-1}(\mathscr{M}), \alpha_{D-1}, \beta^D)$

Their spontaneous breaking gives rise to nontrivial disordered phases

- $\mathscr{S}_{\pi}$ is a non-perturbative tool to identify exotic phases neighboring ordered phases