

Infinite-Order Lattice Anomalies and CPT

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University of Pennsylvania High Energy Theory Seminar



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arXiv:2606.12510

Quantum Systems

Two primary **frameworks** for describing **quantum systems**:

Quantum lattice models (QLMs)*

Quantum field theories (QFTs)

➤ Incredibly **wide-reaching**: cond-mat, hep, quant-ph, & math.

*Discrete space, continuous time

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Different **frameworks**, but intimately related:

$$\{\text{QLMs}\} \xrightarrow{\text{infrared limit}} \{\text{QFTs}\}$$

- Currently, more like a **guiding philosophy**.
- Important to make this relation precise.

*Discrete space, continuous time

Kinematics

A first step is to work at the level of kinematics:

$$\{\text{QLM symmetries}\} \longrightarrow \{\text{QFT symmetries}\}$$

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► Some interesting questions in this **arena** are:

1. What is **{symmetries}** for QLMs and QFTs?
2. Given a QLM **symmetry**, which QFT **symmetries** can it become?
3. Given a QFT **symmetry**, which QLM **symmetries** can it arise from?

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Theme of this talk: question 3 for **invertible 0-form symmetries**.

➤ Which QFT **anomalies** can arise from QLM **anomalies**?

➤ QLMs with Hilbert space $\bigotimes_i \mathcal{H}_i$ and finite $\dim \mathcal{H}_i$

Kinematics

Why care?

Practical QFT reason: A better “lattice laboratory” for QFTs

Practical QLM reason: Leads to new anomalies in QLMs
 $\implies$ new constraints on phases of quantum matter

Conceptual reason: The interplay between QLMs and QFTs continually pushes each other forward.

Theme of this talk: question 3 for invertible 0-form symmetries.

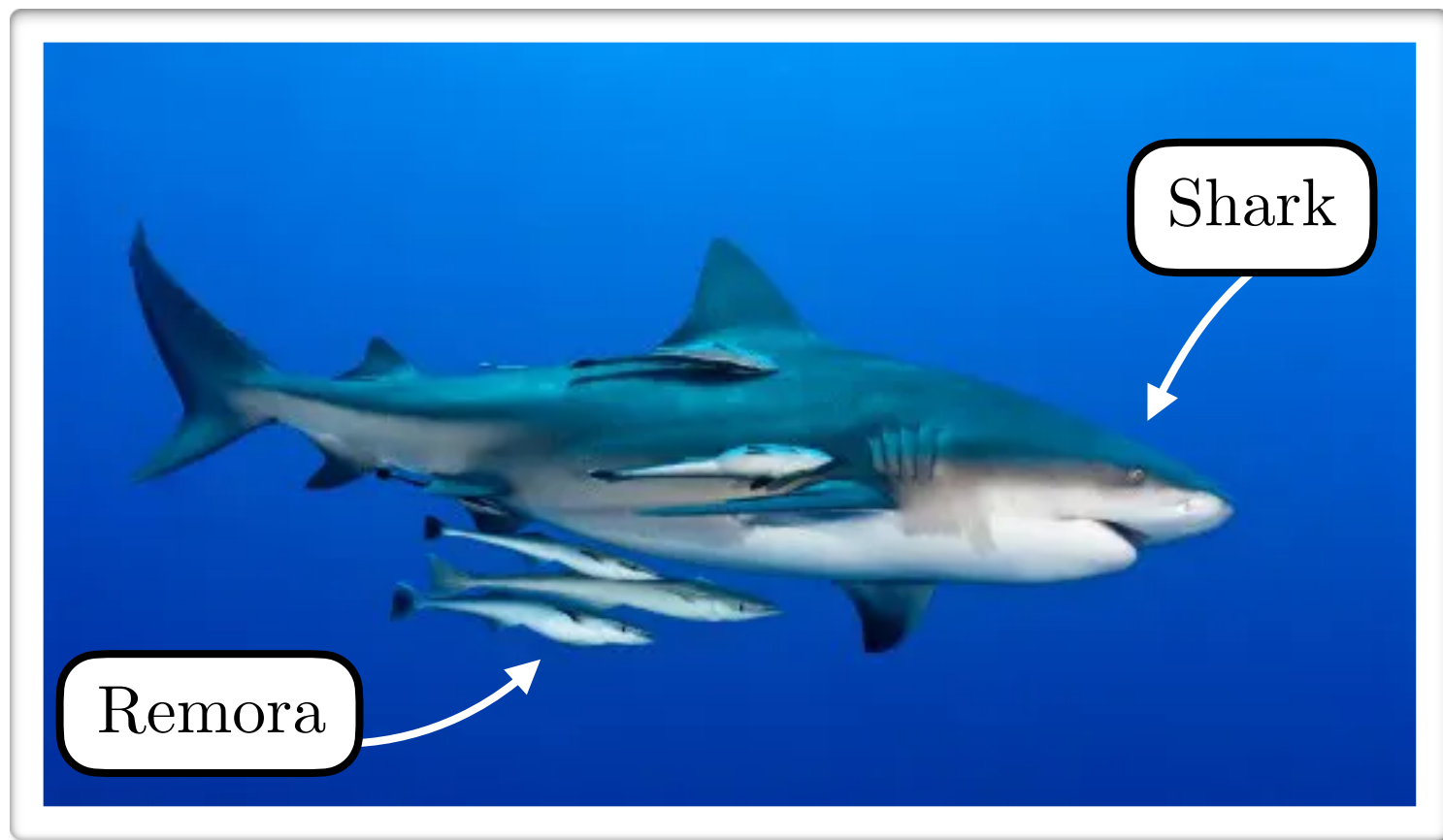
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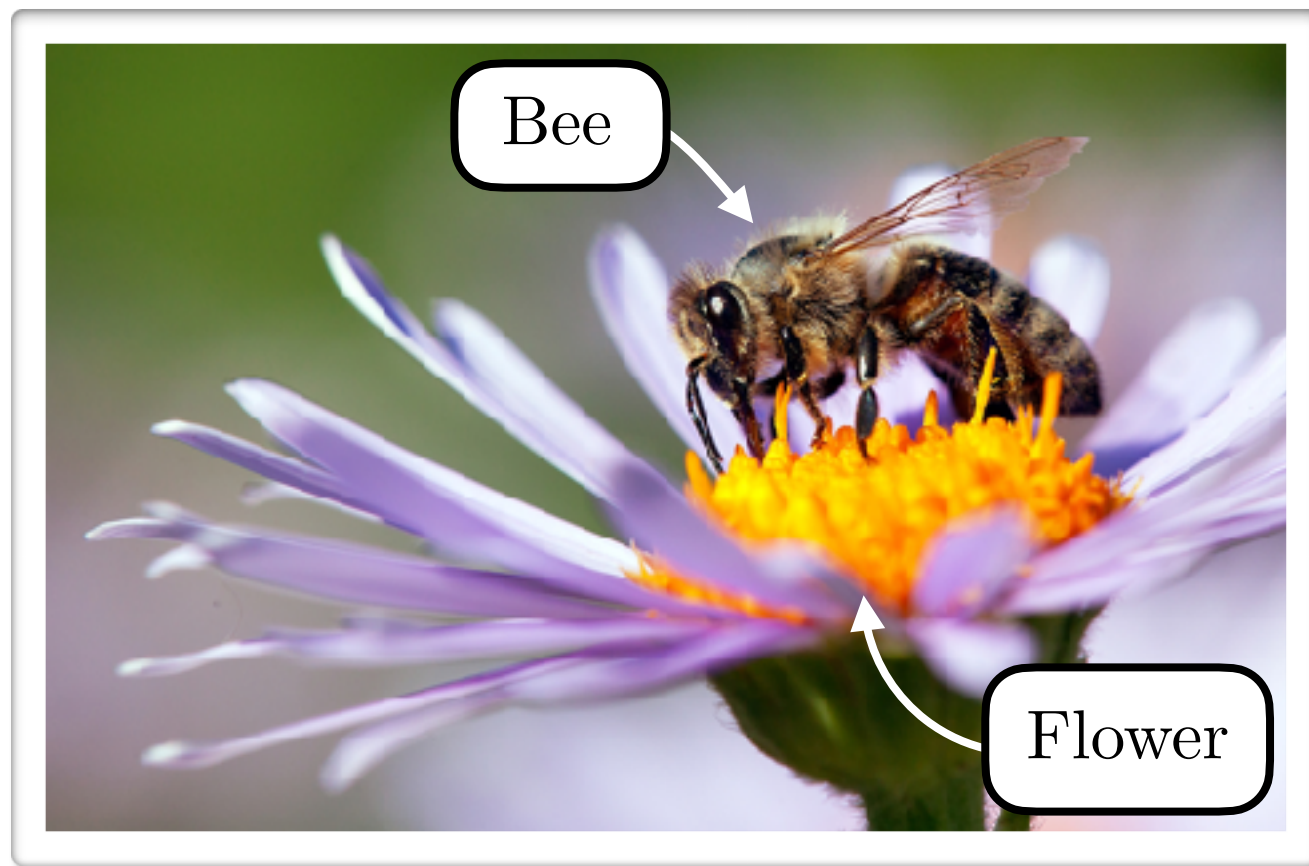


Mutualistic relationship

Kinematics



Kinematics



Kinematics

Knowledge

Outline for this talk

- 1) A light review of **anomalies** of symmetries
- 2) The Onsager symmetry: **Lattice realization** of a 1+1d **Dirac fermion's** vector and axial symmetries
*[Chatterjee, **SP**, Shao '24]*
- 3) **Anomalies** from Onsager + **lattice** CPT symmetry
*[Lew-Smith, **SP**, Shao '26]*



Flower

Anomalies

't Hooft **anomalies** = obstruction* to gauging a global **symmetry**

$$Z[A + df] = Z[A] e^{i\theta(A,f)}$$

➤ Constrains **IR** dynamics: **IR** theory $\neq$ trivial theory.

IR must be gapless, have SSB or topological order, *etc*

➤ Not a binary “yes or no” thing: The same **symmetry** group can have many inequivalent 't Hooft **anomalies**

*Modulo local counterterms

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't Hooft **anomaly** $\neq$ a “**symmetry** of the classical theory fails to be a **symmetry** of the quantum theory” type of **anomaly**

➤ Related to 't Hooft **anomalies** by **gauging**

➤ This talk: **Anomalies** in QFT $\equiv$ 't Hooft **anomalies** in QFT

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$$Z[A + df] = Z[A] e^{i\theta(A,f)}$$

➤ Constrains IR dynamics: IR theory $\neq$ trivial theory

The theory of **anomalies** of ordinary **symmetries** in relativistic QFT is by now a mature subject.

➤ Understanding **anomalies** in QLMs is an increasingly active direction!

[Bols, Carvalho, Chatterjee, Chen, Cheng, Czajka, De Roeck, De Wilde, Else, Feng, Geiko, Hsin, Ji, Kapustin, Kawagoe, Kim, Kobayashi, Levin, Liu, Long, **SP**, Ryu, Seiberg, Seifnashri, Shao, Shirley, Sopenko, Spodyneiko, Thorngren, Tu, Xu, Yi, Zhang, Zou, ...]

➤ Related to 't Hooft **anomalies** by **gauging**

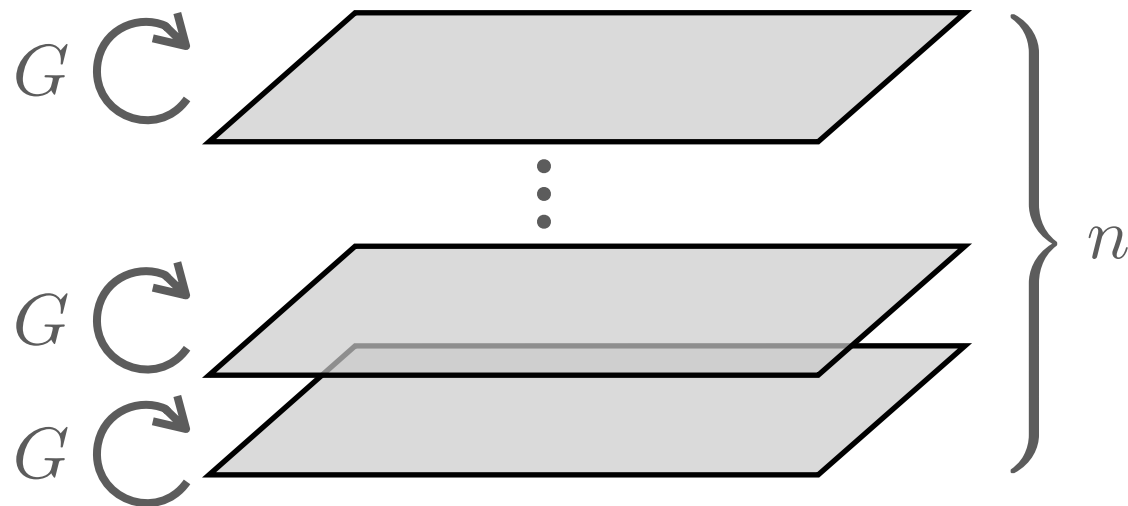
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The Order of an Anomaly

A key characterization of an **anomaly** ω is its **order** $\text{ord}(\omega)$

- Smallest positive integer n such that the diagonal G **symmetry** of the n -copy system is **anomaly-free**.

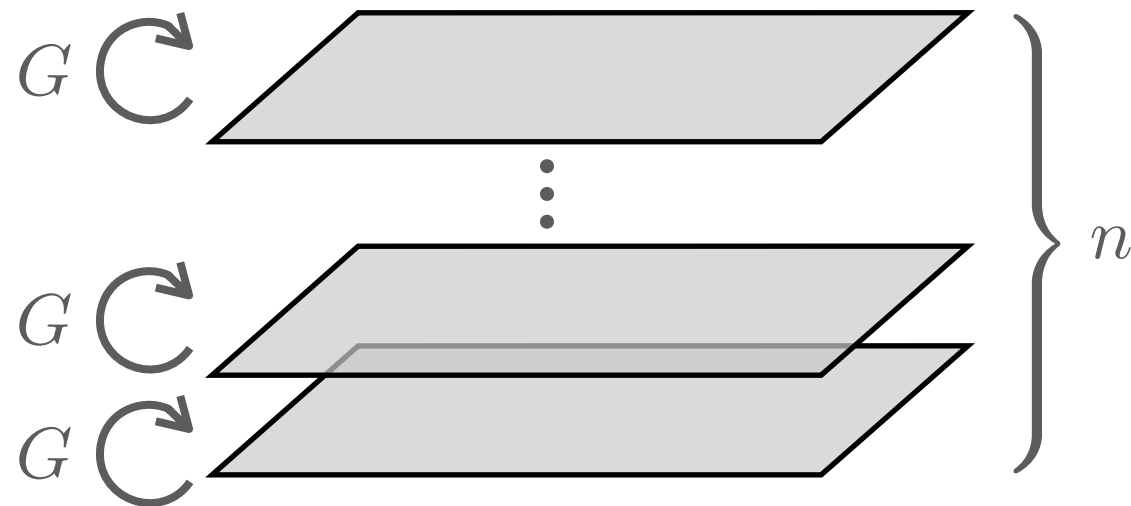


- $\text{ord}(\omega) = 1$ if and only if the G symmetry is anomaly free.

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If $\text{ord}(\omega)$ exists $\implies \omega$ is a **finite-order anomaly**.

- Otherwise, ω is an **infinite-order anomaly**: $\text{ord}(\omega) = \infty$

From the UV to the IR and Back

Given a UV model with G_{UV} **symmetry** and **anomaly** ω_{UV} :

Taking the IR limit induces a **homomorphism**

$$\rho: G_{\text{UV}} \rightarrow G_{\text{IR}}$$

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- In QFT $\implies$ a **homomorphism** $\rho^*: \text{Anom}(G_{\text{IR}}) \rightarrow \text{Anom}(G_{\text{UV}})$ with $\text{Anom}(G)$ the group of inequivalent G **anomalies**.

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Anomaly matching: $\rho^*(\omega_{\text{IR}}) = \omega_{\text{UV}}$

- When the UV and IR symmetries are the same:

$$G_{\text{UV}} \cong G_{\text{IR}} \ \& \ \rho = \text{id} \implies \omega_{\text{IR}} = \omega_{\text{UV}}$$

- A nontrivial UV anomaly implies a nontrivial IR anomaly:

$$\omega_{\text{UV}} = \rho^*(\omega_{\text{IR}}) \neq 1 \implies \omega_{\text{IR}} \neq 1$$

From the UV to the IR and Back

An **anomaly's** **order** can change under the IR limit

➤ The **anomaly matching** condition implies that

$$\text{ord}(\omega_{\text{UV}}) \leq \text{ord}(\omega_{\text{IR}})$$

From the UV to the IR and Back

An **anomaly's order** can change under the IR limit

➤ The **anomaly matching** condition implies that

$$\text{ord}(\omega_{\text{UV}}) \leq \text{ord}(\omega_{\text{IR}})$$

➤ Proof for finite-order IR **anomalies**:

$$\rho^*(\omega_{\text{IR}}) = \omega_{\text{UV}}$$

$$\implies \rho^*(\omega_{\text{IR}})^{\text{ord}(\omega_{\text{IR}})} = \omega_{\text{UV}}^{\text{ord}(\omega_{\text{IR}})}$$

$$\implies 1 = \omega_{\text{UV}}^{\text{ord}(\omega_{\text{IR}})}$$

$$\implies \text{ord}(\omega_{\text{IR}}) = k \text{ord}(\omega_{\text{UV}}) \text{ for some } k \in \mathbb{Z}_{>0}$$

$$\implies \text{ord}(\omega_{\text{UV}}) \leq \text{ord}(\omega_{\text{IR}})$$

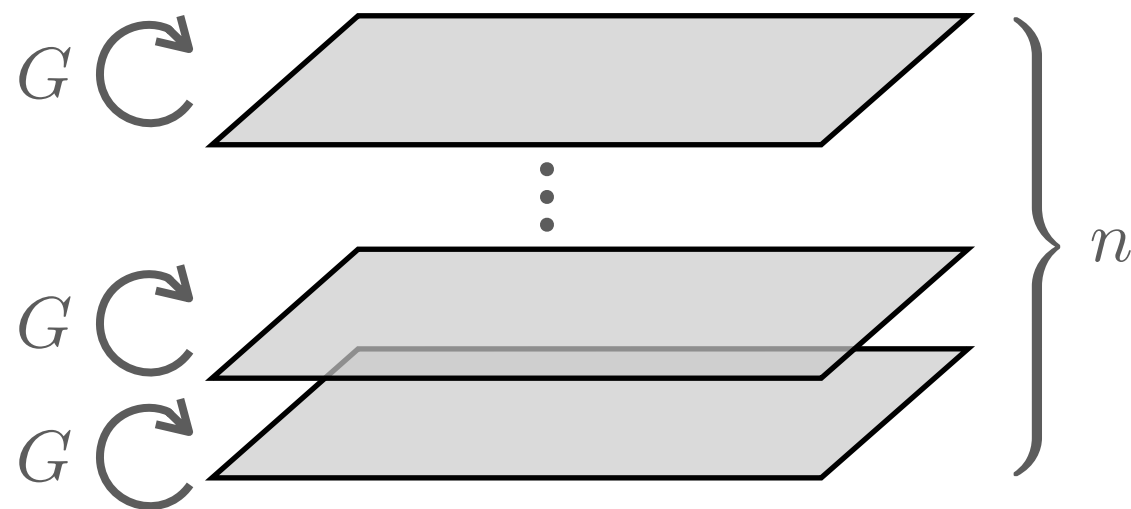
Recap

We just introduced three ideas:

1. An **anomaly** in QFT:

$$Z[A + df] = Z[A] e^{i\theta(A,f)}$$

2. The **order** of an **anomaly**:



3. **Changes** from the UV to the IR:

$$\text{ord}(\omega_{\text{UV}}) \leq \text{ord}(\omega_{\text{IR}})$$

Quantum Mechanics Example

0 + 1d **example**: two qubits with Hamiltonian

$$H = (3 - X_1)(1 + Z_2), \quad \text{Energy spectrum} = \{0, 0, 4, 8\}$$

$G_{UV} = D_8$ generated by $A = CZ_{1,2}$, $B = X_1$, and $C = Z_2$:

$$A^2 = B^2 = C^2 = 1, \quad AB = CBA.$$

► **Anomaly-free**: $\text{ord}(\omega_{UV}) = 1$

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The IR (ground state space) is the $Z_2 = -1$ **eigenspace**

► UV D_8 **symmetry** becomes $\mathbb{Z}_2 \times \mathbb{Z}_2$

$$A_{\text{IR}}^2 = B_{\text{IR}}^2 = 1, \quad A_{\text{IR}}B_{\text{IR}} = -B_{\text{IR}}A_{\text{IR}}$$

► Nontrivial **anomaly**: $\text{ord}(\omega_{\text{IR}}) = 2$

Anomaly inequality: $\text{ord}(\omega_{UV}) = 1 \leq \text{ord}(\omega_{\text{IR}}) = 2$

tl;dr

Infinite-order **anomalies** are elusive in QLMs.*

- No-go theorems for compact locality-preserving **symmetries**
[Kapustin, Sopenko '24; Liu '26]
- In contrast, infinite-order **anomalies** are ubiquitous in QFT
(e.g., perturbative anomalies)

*Of the type we discuss here with locally finite-dimensional, onsite tensor-product Hilbert spaces.

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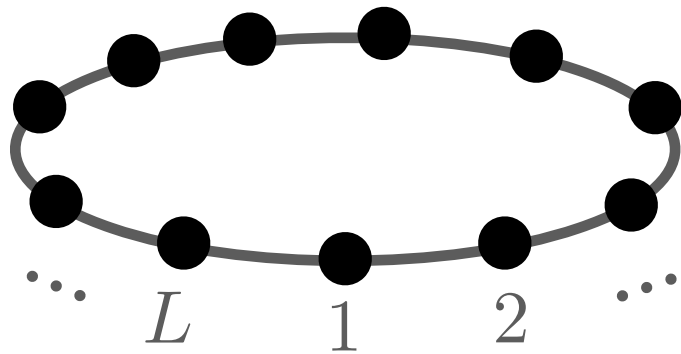
Here, we consider 1 + 1d **fermionic** QLMs and show [Lew-Smith, **SP**, Shao '26]

QLM	UV Symmetry	IR Symmetry	Anomaly Order UV $\rightarrow$ IR
Staggered Fermion	Onsager [Chatterjee, SP , Shao '24]	$\frac{U(1)_V \times U(1)_A}{\mathbb{Z}_2}$	$2 \rightarrow \infty$
	Onsager $\rtimes \mathbb{Z}_2^{\text{CPT}}$	$\frac{U(1)_V \times U(1)_A}{\mathbb{Z}_2} \times \mathbb{Z}_2^{\text{CPT}}$	$\infty \rightarrow \infty$

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A Simple Lattice Model

Consider a **finite lattice** on a ring with even L sites



Fermion operator c_j at each site j

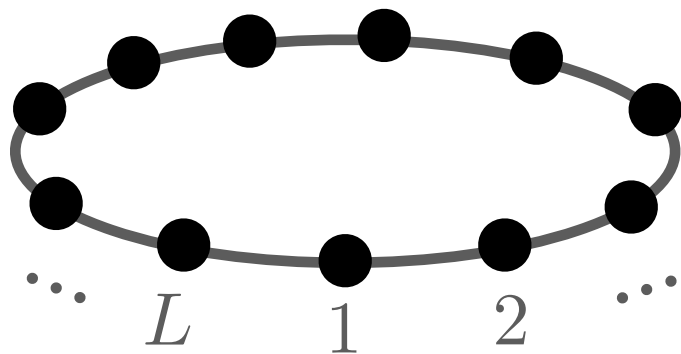
$$\{c_j, c_{j'}^\dagger\} = \delta_{j,j'} \quad \{c_j, c_{j'}\} = 0$$

Staggered fermion Hamiltonian $H = i \sum_{j=1}^L \left(c_j^\dagger c_{j+1} - c_{j+1}^\dagger c_j \right)$

[*Banks, Susskind, Kogut '76*]

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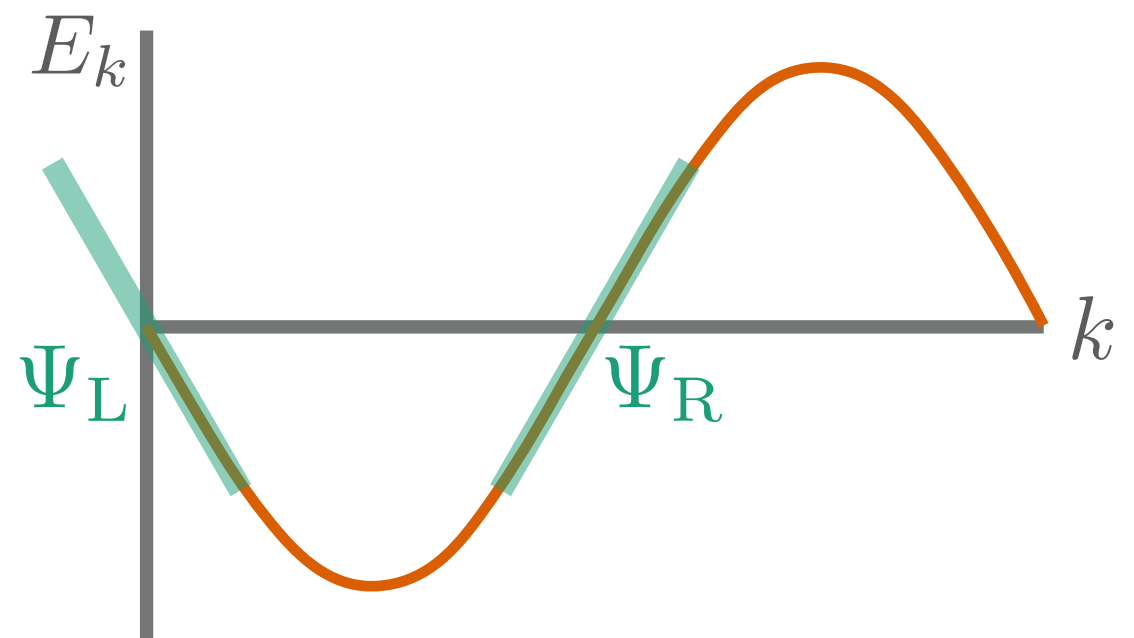
Staggered fermion Hamiltonian $H = i \sum_{j=1}^L \left(c_j^\dagger c_{j+1} - c_{j+1}^\dagger c_j \right)$
[Banks, Susskind, Kogut '76]

➤ H becomes a free, massless **Dirac fermion** theory in IR

➤ In momentum space

$$H = \sum_{k \in \text{BZ}} E_k c_k^\dagger c_k$$

$$E_k = -2 \sin(k)$$



Onsager Symmetry I: What is it

The symmetry of 1+1d massless **Dirac fermion** $\Psi = (\Psi_L \ \Psi_R)^T$

$$U(1)_V : \Psi \mapsto e^{-i\theta_V} \Psi \qquad U(1)_A : \Psi \mapsto e^{-i\theta_A \sigma^z} \Psi$$

Does this **IR symmetry** arise from a **UV symmetry** of H ? Yes!

[Chatterjee, **SP**, Shao '24]

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► **Lattice** vector charge $Q_0 = \sum_{j=1}^L c_j^\dagger c_j$

$$e^{i\theta_0 Q_0} : c_k \mapsto e^{-i\theta_0} c_k$$

► **IR** transformation (look at $k = 0$ and $k = \pi$)

$$\Psi_L \mapsto e^{-i\theta_0} \Psi_L \qquad \Psi_R \mapsto e^{-i\theta_0} \Psi_R$$

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► **Lattice** axial charge $Q_1 = \frac{1}{2} \sum_{j=1}^L (c_j^\dagger c_{j+1} + c_j c_{j+1} + \text{h.c.})$

$$e^{i\theta_1 Q_1} : c_k \mapsto (\cos \theta_1 - i \sin \theta_1 \cos k) c_k - \sin \theta_1 \sin k c_{-k}^\dagger$$

► **IR** transformation (look at $k = 0$ and $k = \pi$)

$$\Psi_L \mapsto e^{-i\theta_1} \Psi_L \qquad \Psi_R \mapsto e^{i\theta_1} \Psi_R$$

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Conserved, quantized charges Q_0 and Q_1 have the **IR** limit

$$Q_0 \xrightarrow{\text{IR}} Q_V$$

$$Q_1 \xrightarrow{\text{IR}} Q_A$$

► Decomposing in terms of real fermions: $c_j = (a_j + ib_j)/2$

$$Q_0 = \frac{i}{2} \sum_{j=1}^L a_j b_j$$

$$Q_1 = \frac{i}{2} \sum_{j=1}^L a_j b_{j+1}$$

Onsager Symmetry I: What is it

These lattice operators do not commute

$$[Q_0, Q_1] \neq 0 \xrightarrow{\text{IR}} [\mathcal{Q}_V, \mathcal{Q}_A] = 0$$

Q_0 and Q_1 generate the **Onsager algebra** [Onsager '44]

$$Q_n = \frac{i}{2} \sum_{j=1}^L a_j b_{j+n} \quad G_n = \frac{i}{2} \sum_{j=1}^L (a_j a_{j+n} - b_j b_{j+n})$$

$$[Q_n, Q_m] = iG_{m-n} \quad [G_n, G_m] = 0$$

$$[Q_n, G_m] = 2i(Q_{n-m} - Q_{n+m})$$

UV to **IR** symmetry map

$$Q_n \xrightarrow{\text{IR}} \begin{cases} \mathcal{Q}_V & n \text{ even} \\ \mathcal{Q}_A & n \text{ odd} \end{cases} \quad G_n \xrightarrow{\text{IR}} 0$$

Onsager Symmetry I: What is it

Remarks on this Onsager symmetry

- It is not part of the **free-fermion** symmetry

$$c_k \mapsto e^{i\phi(k)} c_k$$

- It is a **non-internal** symmetry. E.g.,

$$e^{-i\frac{\pi}{2}Q_1} e^{i\frac{\pi}{2}Q_0} : (a_j, b_j) \mapsto (a_{j-1}, b_{j+1})$$

- Its symmetry operators are **locality-preserving** but generically not quantum cellular automata. E.g.,

$$e^{i\lambda(Q_{-1}+Q_1)}$$

- Non-compact symmetry group: includes $\mathbb{Z}$ and $\mathbb{R}$ subgroups

Onsager Symmetry II: Its Anomaly.....

The Dirac fermion's $[U(1)_V \times U(1)_A] / \mathbb{Z}_2$ symmetry is anomalous

► $S_{\text{inflow}} = \frac{i}{\pi} \int A_V dA_A \implies$ infinite-order anomaly

Is the Onsager symmetry anomalous?

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Is the Onsager symmetry anomalous?

What even is an anomaly in a QLM?

Here, we view an anomaly as a robust obstruction to a trivial* symmetric gapped phase

- Avoids having to define gauging
- Generalization of the Lieb-Schultz-Mattis theorem
- Commonly used for non-invertible symmetry

*Trivial meaning without discrete symmetry-breaking/topological order

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Is the Onsager symmetry anomalous? Yes! [*Chatterjee, SP, Shao '24*]

➤ Can prove each symmetric local Hamiltonian takes the form

$$H_g = i \sum_n \sum_{j=1}^L g_n (a_j a_{j+n} + b_j b_{j+n}),$$

which never has a unique gapped ground state

➤ Each $H_g \neq 0$ is gapless $\implies$ symmetry-enforced gaplessness

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In the staggered fermion Hamiltonian, the **IR Dirac fermion's** **anomaly** arises from the **Onsager** symmetry's **lattice anomaly**.

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Remarks

- Not all Onsager symmetries are anomalous
- For this Onsager symmetry, there are non-quadratic but terribly non-local symmetric Hamiltonians
- Not all Onsager symmetries disallow local non-quadratic terms [*Vernier, O'Brien, Fendley '18 ; SP, Chatterjee, Shao '24*]

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Onsager Symmetry III: Anomaly Order

What is the order of the **Onsager** symmetry's **anomaly**?

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It is **order two**: The diagonal **Onsager** symmetry of the 2-copy system is **anomaly-free** [*Lew-Smith, **SP**, Shao '26*]

Onsager Symmetry III: Anomaly Order

What is the order of the **Onsager** symmetry's **anomaly**?

It is **order two**: The diagonal **Onsager** symmetry of the 2-copy system is **anomaly-free** [Lew-Smith, *SP*, Shao '26]

➤ **2-copy** system has two fermion operators $c_j^{\text{I}}, c_j^{\text{II}}$ per site j .

➤ Diagonal **Onsager** symmetry generated by

$$Q_0^{\text{diag}} = Q_0^{\text{I}} + Q_0^{\text{II}}, \quad Q_1^{\text{diag}} = Q_1^{\text{I}} + Q_1^{\text{II}}$$

➤ A symmetric Hamiltonian with a unique gapped ground state:

$$H_{\text{gap}} = \frac{i}{2} \sum_{j=1}^L (a_j^{\text{I}} a_j^{\text{II}} + b_j^{\text{I}} b_j^{\text{II}})$$

$\Rightarrow$ The $(Q_0^{\text{diag}}, Q_1^{\text{diag}})$ **Onsager** symmetry is **anomaly-free**

UV to IR Anomaly Order

QLM	UV Symmetry	IR Symmetry	Anomaly Order UV $\rightarrow$ IR
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Deformed 2-copy staggered fermion Hamiltonian

$$\frac{i}{2} \sum_{j=1}^L (a_j^{\text{I}} a_{j+1}^{\text{I}} + b_j^{\text{I}} b_{j+1}^{\text{I}} + a_j^{\text{II}} a_{j+1}^{\text{II}} + b_j^{\text{II}} b_{j+1}^{\text{II}}) + g H_{\text{gap}}$$

- Single-particle dispersions $E_k^{\pm} = -2 \sin(k) \pm g$
- Trivially gapped for $|g| > 2$

IR limit of H_{gap} deformation ($g \ll 1$): chemical potential term

$$i\mu(\Psi_{\text{L}}^{\text{I}})^{\dagger} \Psi_{\text{L}}^{\text{II}} + i\mu(\Psi_{\text{R}}^{\text{I}})^{\dagger} \Psi_{\text{R}}^{\text{II}} + \text{h.c.}$$

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Can the order of the **Onsager** symmetry's **anomaly** be enhanced to infinite order?

- One way to increase an **anomaly's** order is by introducing additional symmetry

Here, we will add a **lattice CPT symmetry**

- Natural symmetry to add from the QFT perspective

Lattice CPT Symmetry

Consider the anti-unitary operator Π satisfying

$$\Pi^2 = 1, \quad \Pi i \Pi^\dagger = -i, \quad \Pi c_j \Pi^\dagger = c_{-j+1}^\dagger$$

► Is a $\mathbb{Z}_2$ anti-unitary reflection operator

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► Is a $\mathbb{Z}_2$ **anti-unitary reflection** operator

It is a symmetry of the **staggered fermion Hamiltonian**

$$H = i \sum_{j=1}^L \left(c_j^\dagger c_{j+1} - c_{j+1}^\dagger c_j \right)$$

► In the IR, Π becomes the **CPT** symmetry operator of the **Dirac fermion** QFT

$\implies \Pi$ is a lattice $\mathbb{Z}_2^{\text{CPT}}$ symmetry operator

Lattice CPT Symmetry

Consider the anti-unitary operator Π satisfying

Π acts on the Onsager charge operators as

$$\Pi Q_n \Pi^\dagger = -Q_{-n}, \quad \Pi G_n \Pi^\dagger = G_n$$

(Q_0, Q_1, Π) form the symmetry group

$$\text{Onsager} \rtimes \mathbb{Z}_2^{\text{CPT}}$$

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In the IR of the staggered fermion Hamiltonian

$$\Pi Q_n \Pi^\dagger = -Q_{-n} \xrightarrow{\text{IR}} \begin{cases} \Pi_{\text{IR}} Q_V \Pi_{\text{IR}}^\dagger = -Q_V & n \text{ even} \\ \Pi_{\text{IR}} Q_A \Pi_{\text{IR}}^\dagger = -Q_A & n \text{ odd} \end{cases}$$

$$\text{Onsager} \rtimes \mathbb{Z}_2^{\text{CPT}} \xrightarrow{\text{IR}} [\text{U}(1)_V \times \text{U}(1)_A] / \mathbb{Z}_2 \times \mathbb{Z}_2^{\text{CPT}}$$

Onsager + CPT: Anomaly Order

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➤ **N -copy** system has N fermion operators $\{c_j^{(\ell)}\}_{\ell=1}^N$ per site j

➤ Diagonal Onsager $\rtimes \mathbb{Z}_2^{\text{CPT}}$ symmetry generated by

$$Q_0^{\text{diag}} = \sum_{\ell=1}^N Q_0^{(\ell)}, \quad Q_1^{\text{diag}} = \sum_{\ell=1}^N Q_1^{(\ell)}, \quad \Pi^{\text{diag}}$$

➤ Can prove each symmetric local Hamiltonian takes the form

$$\frac{i}{2} \sum_n \sum_{I,J=1}^N \sum_{j=1}^L g_n^{IJ} (a_j^I a_{j+n}^J + b_j^I b_{j+n}^J) \quad (g_{-n}^{IJ} = -g_n^{IJ})$$

which never has a unique gapped ground state

Recap

QLM	UV Symmetry	IR Symmetry	Anomaly Order UV $\rightarrow$ IR
Staggered Fermion	Onsager	$\frac{U(1)_V \times U(1)_A}{\mathbb{Z}_2}$	$2 \rightarrow \infty$

Recap

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Heisenberg Chain	$\mathbb{Z}^{\text{tran}}$	$\mathbb{Z}_2$	$1 \rightarrow 2$ <i>[Metlitski, Thorngren '17]</i>
	$\mathbb{Z}^{\text{tran}} \rtimes \mathbb{Z}_2^{\text{CPT}}$	$\mathbb{Z}_2 \times \mathbb{Z}_2^{\text{CPT}}$	$2 \rightarrow 2$ <i>[Seiberg, Shao, Zhang '25]</i>
Kitaev Chain	$\mathbb{Z}^{\text{maj}}$	$\mathbb{Z}_2$	$2 \rightarrow 8$ <i>[Seiberg, Shao '23]</i>
	$\mathbb{Z}^{\text{maj}} \rtimes \mathbb{Z}_2^{\text{CPT}}$	$\mathbb{Z}_2 \times \mathbb{Z}_2^{\text{CPT}}$	$8 \rightarrow 8$ <i>[Seiberg, Shao, Zhang '25]</i>

Future Directions

The role of **CPT**, and more generally **CRT**, in QLMs

- Interesting results also noted for invertible phases [*Ogata '22; Sopenko '25*]

Systematic study of the **anomaly's** stability to ancillas

- Preliminary results: the **anomaly** is robust to a natural class of non-onsite ancillas [*Lew-Smith, SP, Shao '26*]

Analogous story for **Onsager** symmetries in $> 1 + 1d$

- The **anomalies** of such **Onsager** symmetries can realize parity anomalies [*SP, Kim, Chatterjee, Shao '25*], Witten's $SU(2)$ anomaly [*Gioia, Thorngren '25*], and symmetry-enforced Fermi surfaces [*Kim, SP, Shao '25*].

Backup Slides

Anomalies in quantum mechanics

Consider QM model with Hilbert space $\mathcal{H}$ and Hamiltonian H

- Assume there is a unitary G symmetry: $[U_g, H] = 0, g \in G$
- Symmetry has an anomaly if

$$U_g U_h = e^{i\theta(g,h)} U_{gh} \quad e^{i\theta(g,h)} \neq e^{i(f(g)+f(h)-f(gh))}$$

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Proof:

1. Assume $|\text{gs}\rangle \in \mathcal{H}$ is a unique gapped ground state of H
2. Therefore, $U_g U_h |\text{gs}\rangle = e^{if(h)} U_g |\text{gs}\rangle = e^{i(f(g)+f(h))} |\text{gs}\rangle$
3. However, $U_g U_h |\text{gs}\rangle = e^{i\theta(g,h)} U_{gh} |\text{gs}\rangle = e^{i(\theta(g,h)+f(gh))} |\text{gs}\rangle$
4. Requires $e^{i\theta(g,h)} = e^{i(f(g)+f(h)-f(gh))} \implies \underline{\text{contradiction}}$

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Example:

- A qubit, $\mathcal{H} = \text{span}_{\mathbb{C}}\{|\uparrow\rangle, |\downarrow\rangle\}$, with $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry

$$U_{(n,m)} = X^n Z^m \quad (n, m) \in \mathbb{Z}_2 \times \mathbb{Z}_2$$

- $U_{(n,m)} U_{(n',m')} = (-1)^{mn'} U_{(n+n',m+m')} \implies \text{anomaly}$

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- Explicit check: $[U_{(n,m)}, H] = 0 \implies H \propto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$