Realizing the chiral anomaly on the lattice

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UMN theory seminar

Arkya
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arXiv:2409.12220 [Phys. Rev. Lett.]

What do things do?

One of the most elementary questions in the quantum physics of many degrees of freedom:

- Given a fixed microscopic (UV) set up, which macroscopic (IR) phenomena can arise?
- Central to various areas of cond-mat, hep-th, and math-ph

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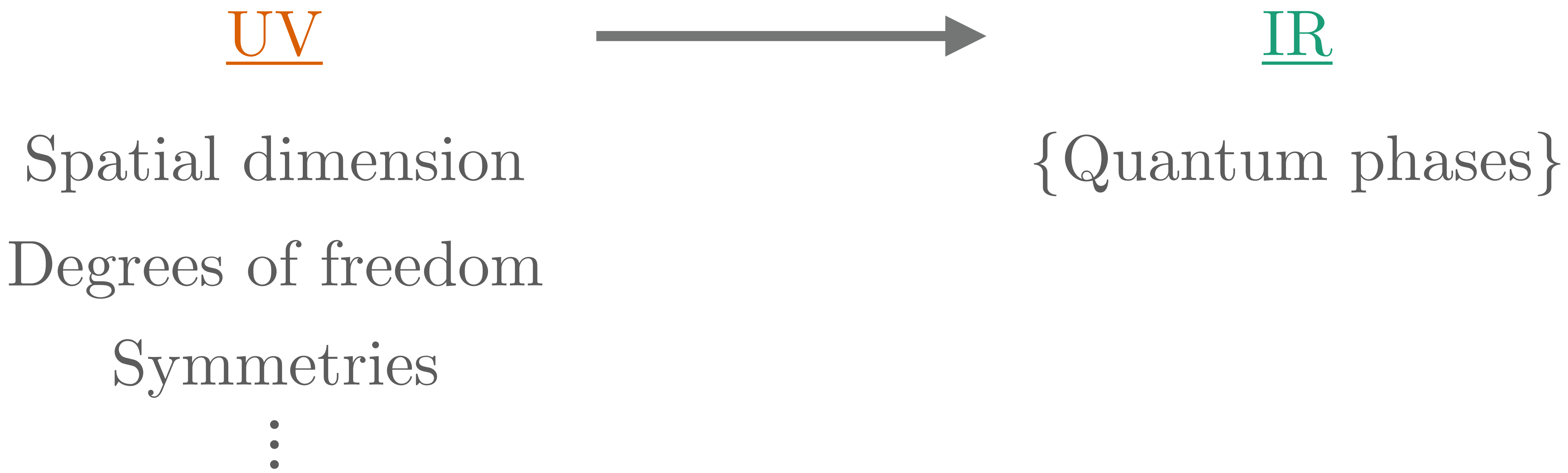
Prototypical example: quantum phases of matter



UV

Spatial dimension

Degrees of freedom

Symmetries

⋮

IR

{Quantum phases}

A first organization of quantum phases

The set {Quantum phases} is generally *very* complicated

➤ Partition it based on the phases' low-energy spectra

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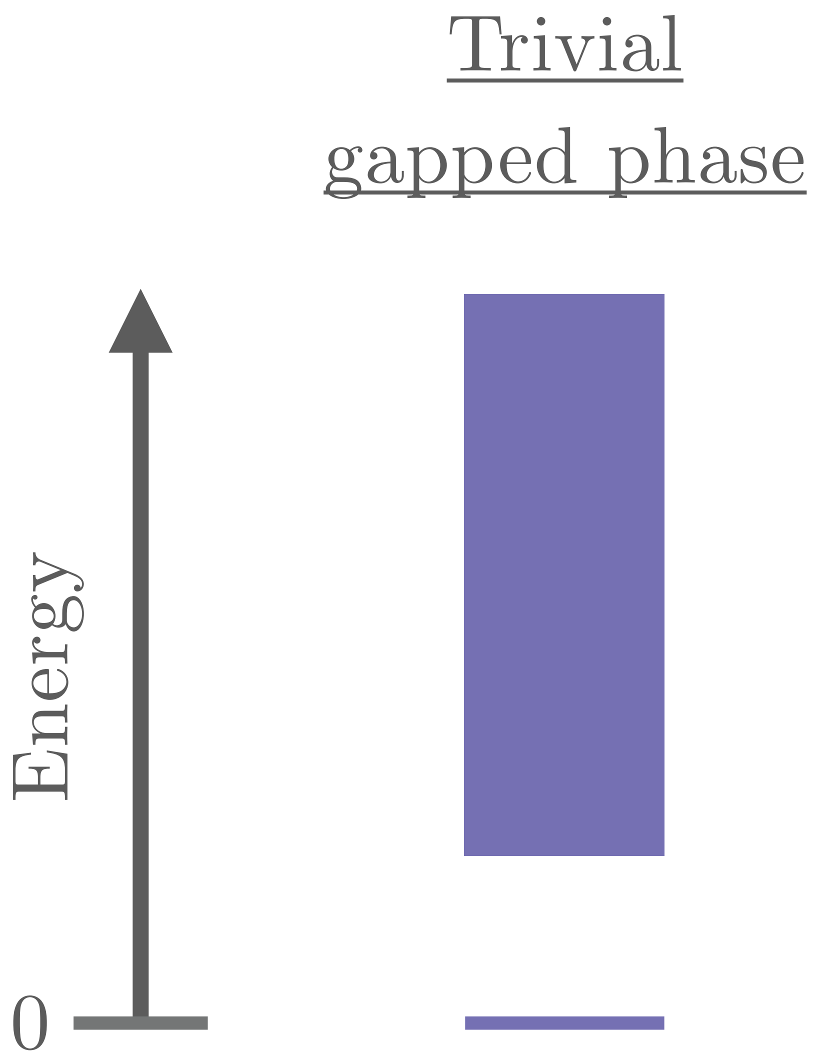

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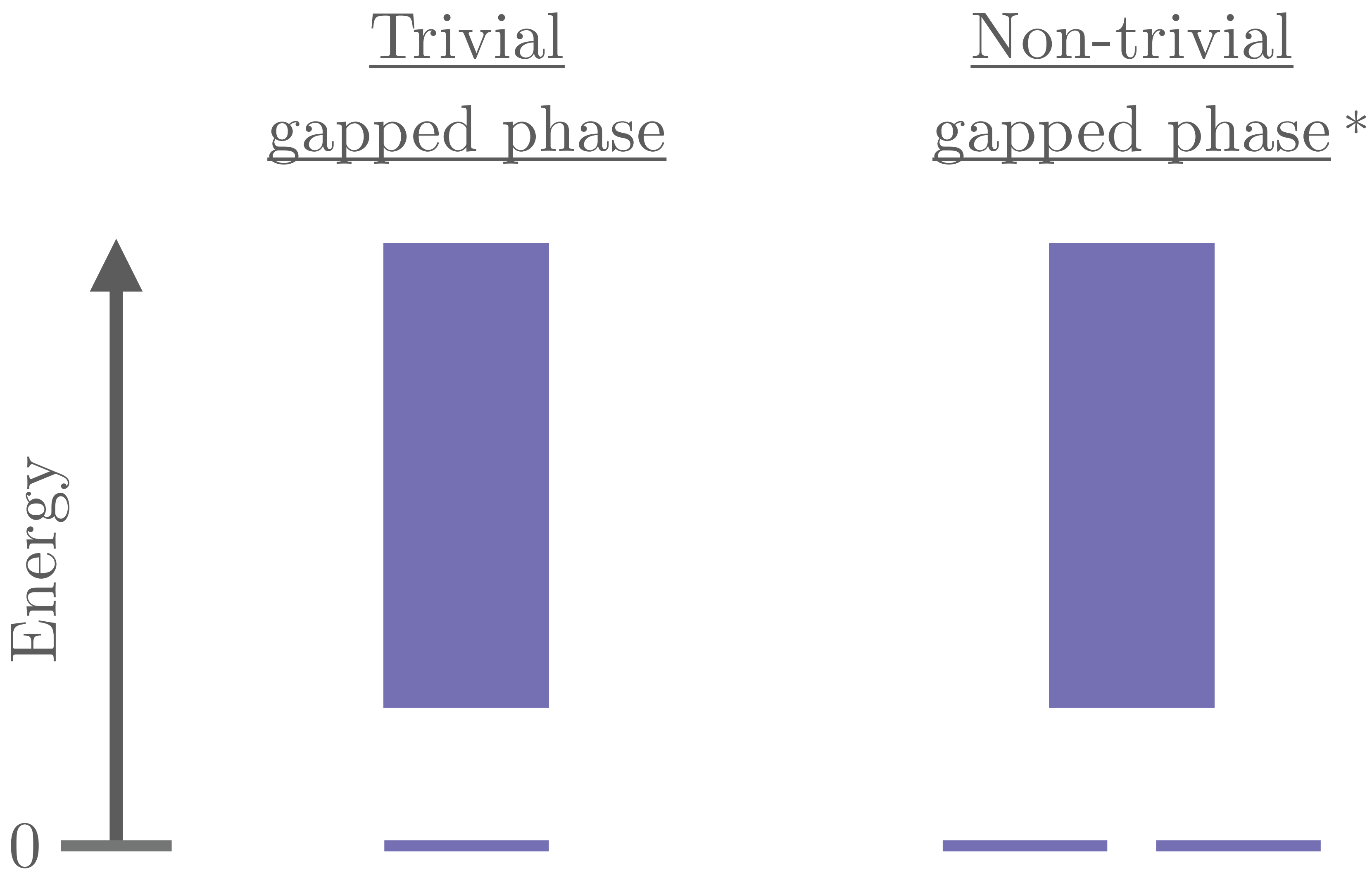

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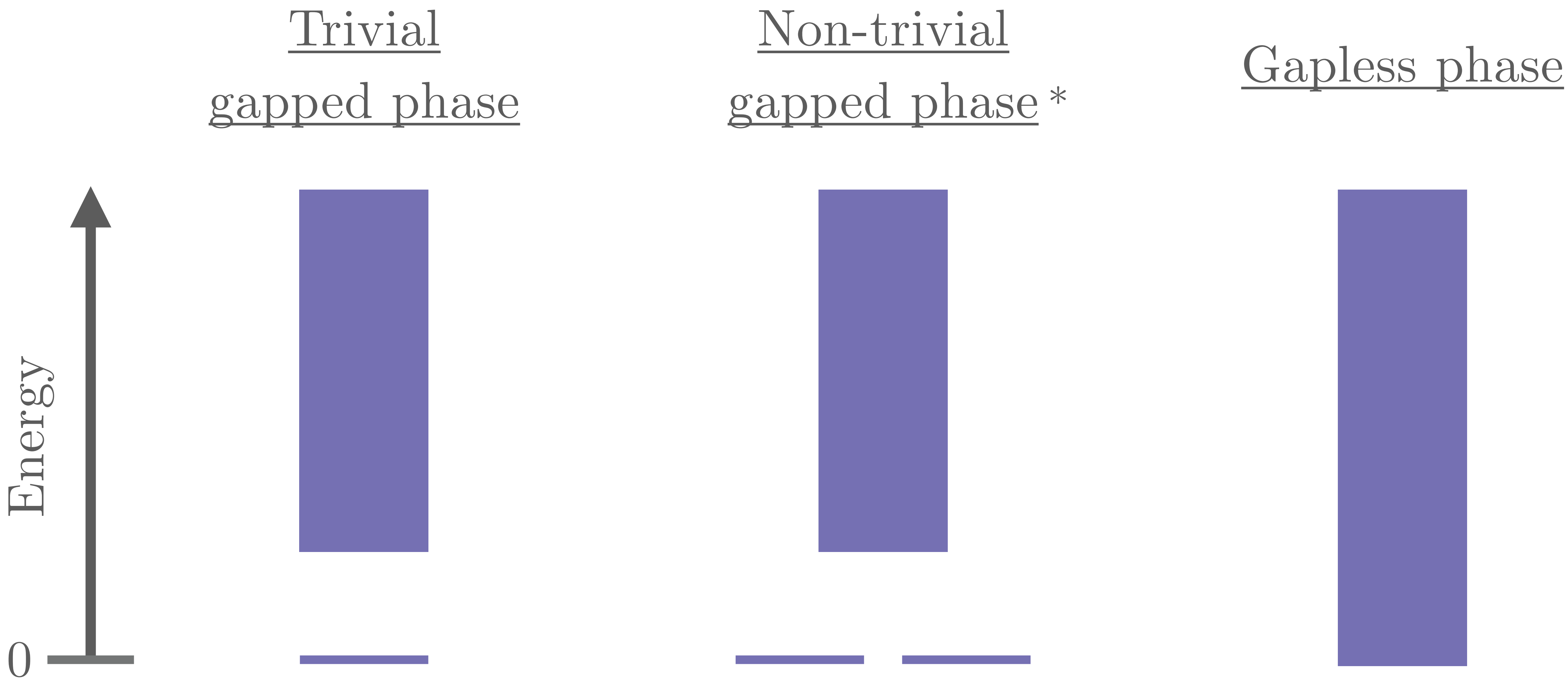

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Symmetry constrains

UV symmetries can classify and characterize phases

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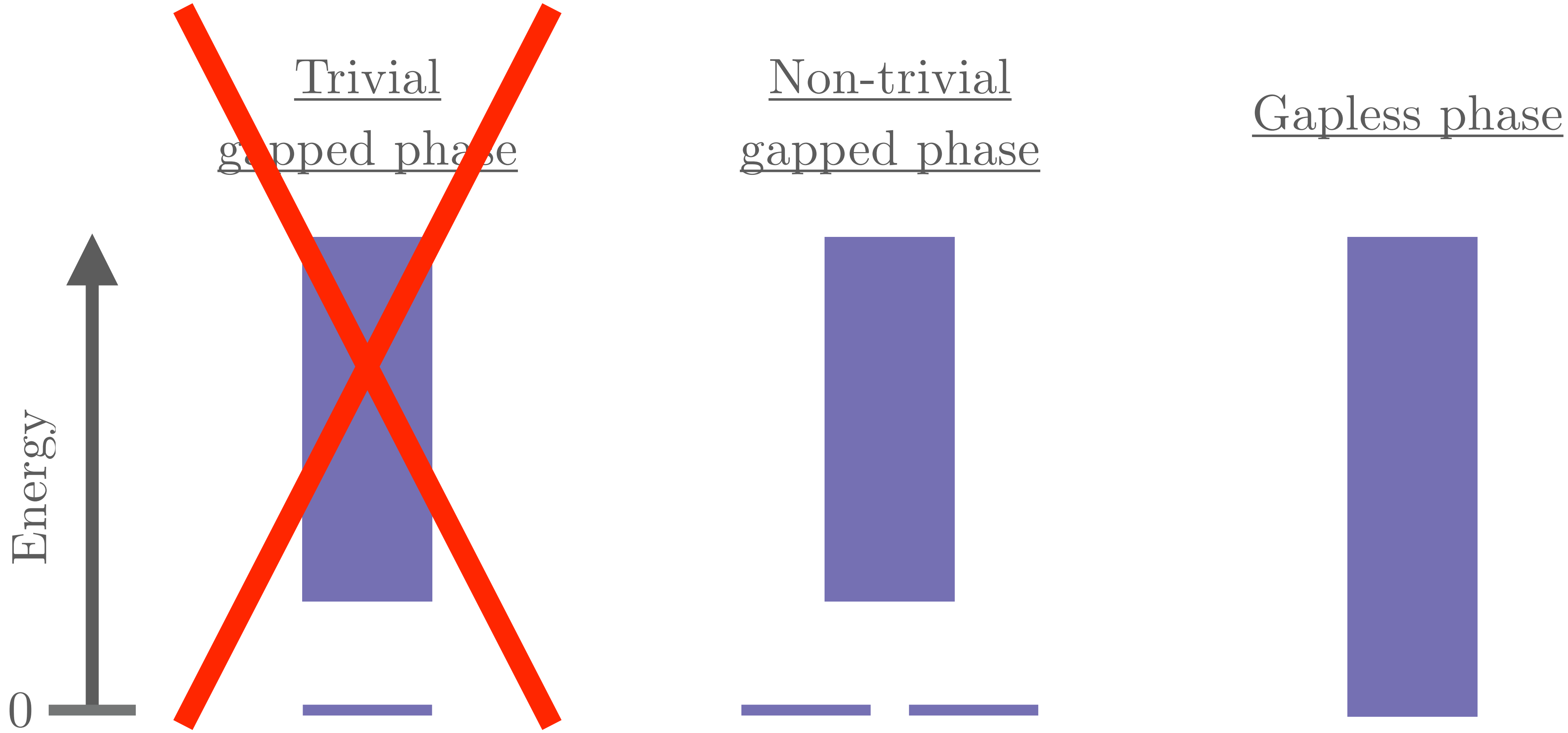
Global symmetries can also be incompatible with trivial gapped symmetric phases.



- ➤ Every phase has SSB/topological order, is gapless, *etc.*

Anomalies

Symmetry with no trivial gapped phases = Anomalous

Anomalies

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Important disclaimers (a.k.a., please, don't sue me)

1. This is different from the "classical symmetry fails to be a quantum symmetry" type anomaly
2. It is a generalization of 't Hooft anomalies, which are obstructions to gauging a symmetry.
3. It is an increasing popular perspective for anomalous spacetime and generalized symmetries

[C.-M. Chang, Y.-H. Lin, S.-H. Shao, Y. Wang, X. Yin '18; X.-G. Wen '18; R. Thorngren, Y. Wang '19; Y. Choi, C. Córdova, P.-S. Hsin, H.T. Lam, S.-H. Shao '21; ⋯ ; W. Shirley, C. Zhang, W. Ji, M. Levin '25]

Anomalies in quantum mechanics

Consider QM model with Hilbert space $\mathscr{H}$ and Hamiltonian H

- Assume there is a unitary G symmetry: $[U_g, H] = 0\,,\ g \in G$
- Symmetry has an anomaly if

$$U_g U_h = \mathrm{e}^{\mathrm{i}\theta(g,h)} U_{gh} \qquad \mathrm{e}^{\mathrm{i}\theta(g,h)} \neq \mathrm{e}^{\mathrm{i}(f(g)+f(h)-f(gh))}$$

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Proof:

1. Assume $|\mathrm{gs}\rangle \in \mathscr{H}$ is a unique gapped ground state of H
2. Therefore, $U_g U_h |\mathrm{gs}\rangle = \mathrm{e}^{\mathrm{i}f(h)} U_g |\mathrm{gs}\rangle = \mathrm{e}^{\mathrm{i}(f(g)+f(h))} |\mathrm{gs}\rangle$
3. However, $U_g U_h |\mathrm{gs}\rangle = \mathrm{e}^{\mathrm{i}\theta(g,h)} U_{gh} |\mathrm{gs}\rangle = \mathrm{e}^{\mathrm{i}(\theta(g,h)+f(gh))} |\mathrm{gs}\rangle$
4. Requires $\mathrm{e}^{\mathrm{i}\theta(g,h)} = \mathrm{e}^{\mathrm{i}(f(g)+f(h)-f(gh))} \Longrightarrow$ contradiction

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Example:

- A qubit, $\mathscr{H} = \mathrm{span}_{\mathbb{C}}\{|\uparrow\rangle, |\downarrow\rangle\}$, with $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry

$$U_{(n,m)} = X^n Z^m \qquad (n,m) \in \mathbb{Z}_2 \times \mathbb{Z}_2$$

- $U_{(n,m)} U_{(n',m')} = (-1)^{mn'} U_{(n+n',m+m')} \implies$ anomaly

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Anomalies in >(0+1)D are much richer

- Include the anomalies from QM (i.e., projective representations) and much more due to locality.

Ex

- A y

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Emergent vs emanant

UV model

symmetries

anomalies

Low energy

limit

IR model

symmetries

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The symmetries and anomalies in the IR are generally different from those in the UV

Emergent vs emanant

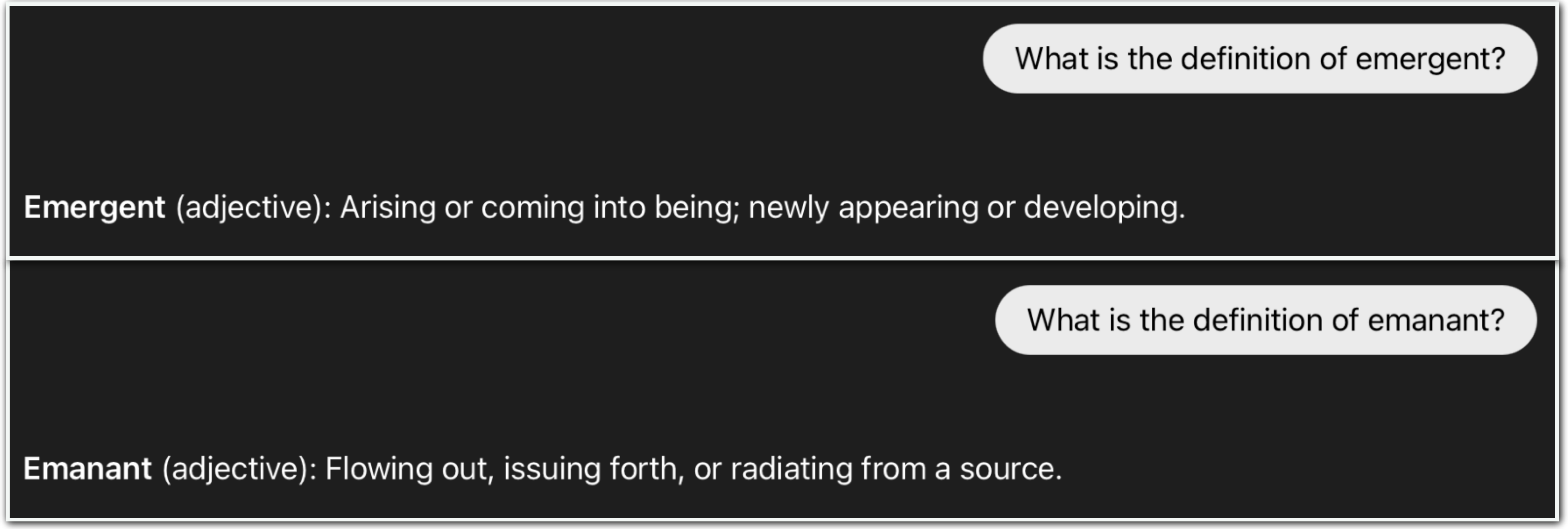
The symmetries and anomalies in the IR are generally different from those in the UV

- For a given UV model, there are two types of IR symmetries and anomalies:

  1) Emergent (accidental): have no UV counterpart

  2) Emanant: have a UV counterpart [M. Cheng, N. Seiberg '22]

Emergent vs emanant



What is the definition of emergent?

Emergent (adjective): Arising or coming into being; newly appearing or developing.

What is the definition of emanant?

Emanant (adjective): Flowing out, issuing forth, or radiating from a source.

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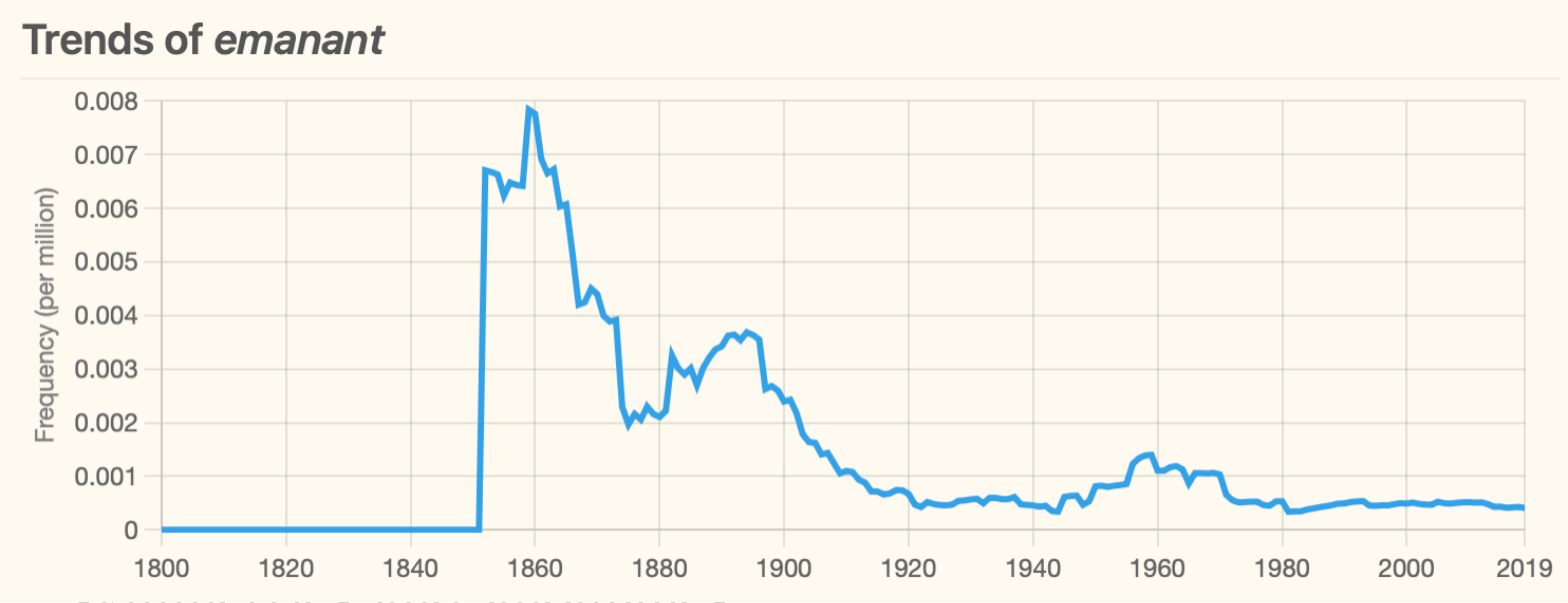

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Emergent vs emanant

<u>Five possibilities</u>

Emergent symmetry with no anomaly

Emergent symmetry with emergent anomaly

Emanant symmetry with no anomaly

Emanant symmetry with emanant anomaly

Emanant symmetry with emergent anomaly

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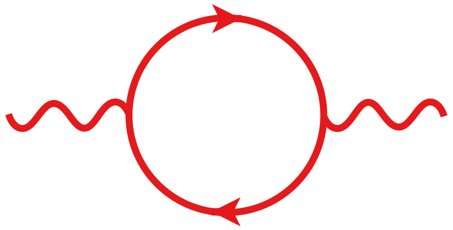
Lattice vs QFT anomalies

Often, the UV model is a lattice model

Lattice model $\xrightarrow[\text{limit}]{\text{Low energy}}$ QFT

Many QFT anomalies cannot be realized exactly on the lattice.*

- For example: perturbative anomalies
  [A. Kapustin, N. Sopenko '24; Y.-T. Tu, D. Long, D. Else '25; R. Liu '26]



**with onsite tensor factorized Hilbert space and finite dimensional local Hilbert spaces*

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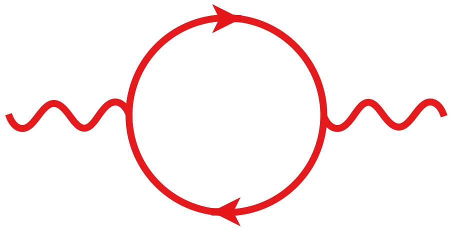

Can perturbative anomalies emanate from a lattice anomaly?

- Punchline for this talk: Yes, the chiral anomaly can!
- Rest of talk: demonstrating this in a simple lattice model

---

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Why care?

Practical reason: Better UV symmetries means a better "lattice laboratory" for QFTs (*eg*, chiral gauge theory)

Conceptual reason I: New lattice anomalies

Conceptual reason II: The interplay between lattice models and QFTs continually push each other forward.

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Lattice vs QFT anomalies

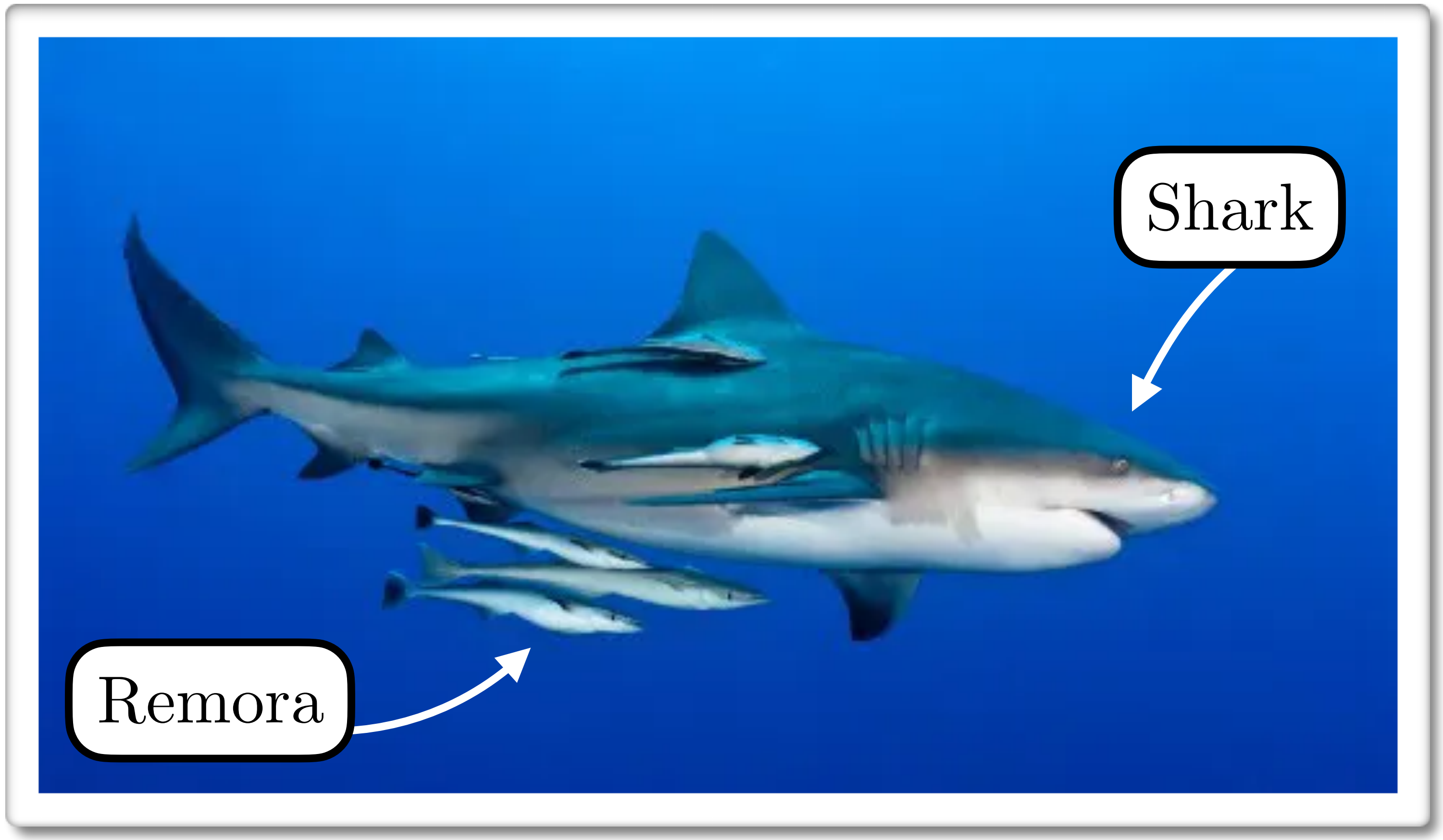
Mutualistic relationship

Lattice vs QFT anomalies

Knowledge

Lattice

Continuum



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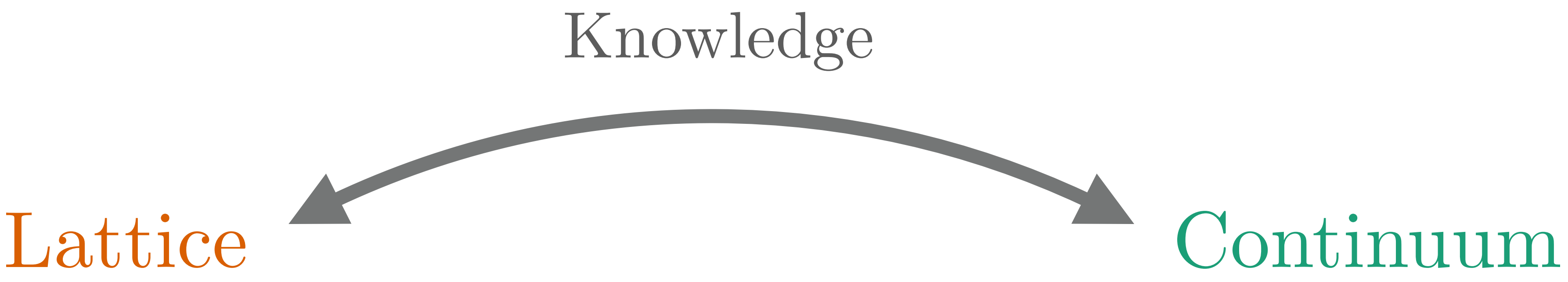

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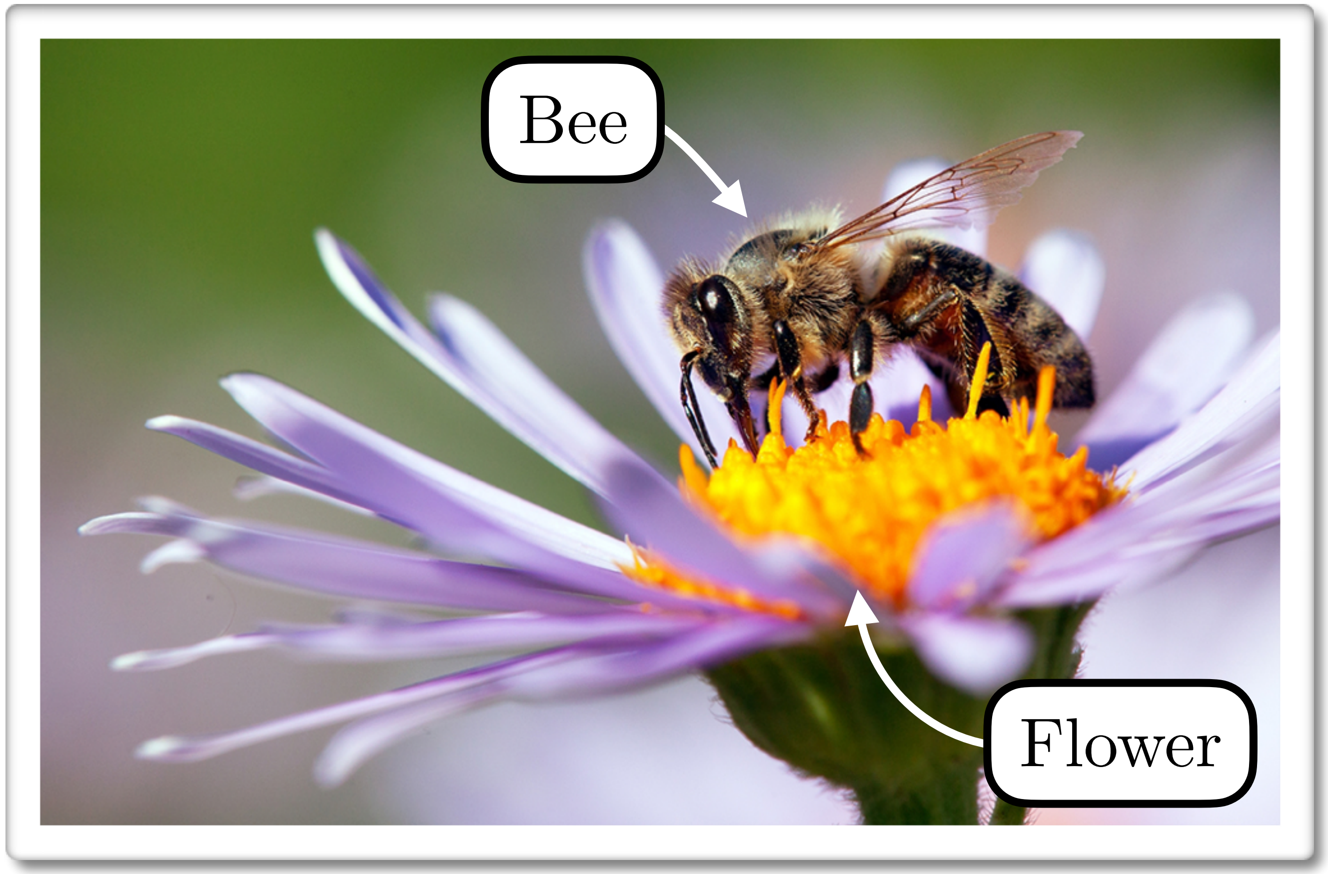

Dirac fermion field theory

Free, massless Dirac fermion $\Psi = (\Psi_\mathrm{L}, \Psi_\mathrm{R})^\mathsf{T}$ in $1+1$D:

$$\mathscr{L} = \mathrm{i}\,\Psi_\mathrm{L}^\dagger(\partial_t + \partial_x)\Psi_\mathrm{L} + \mathrm{i}\,\Psi_\mathrm{R}^\dagger(\partial_t - \partial_x)\Psi_\mathrm{R}$$

- Ψ_L (Ψ_R) is a left (right) moving complex Weyl fermion

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Chiral $(\mathrm{U}(1)^V \times \mathrm{U}(1)^A)/\mathbb{Z}_2$ symmetry

$$\begin{aligned} &\text{vector } \mathrm{U}(1)^V: \quad && \Psi_{\mathrm{L}}^{\dagger} \mapsto \mathrm{e}^{+\mathrm{i}\theta}\Psi_{\mathrm{L}}^{\dagger} \qquad && \Psi_{\mathrm{R}}^{\dagger} \mapsto \mathrm{e}^{+\mathrm{i}\theta}\Psi_{\mathrm{R}}^{\dagger} \\ &\text{axial } \mathrm{U}(1)^A: \quad && \Psi_{\mathrm{L}}^{\dagger} \mapsto \mathrm{e}^{+\mathrm{i}\alpha}\Psi_{\mathrm{L}}^{\dagger} \qquad && \Psi_{\mathrm{R}}^{\dagger} \mapsto \mathrm{e}^{-\mathrm{i}\alpha}\Psi_{\mathrm{R}}^{\dagger} \end{aligned}$$

➤ Axial charge $Q^A = C_R Q^V C_R^{\dagger}$, where $C_R \colon \Psi_{\mathrm{R}} \mapsto \Psi_{\mathrm{R}}^{\dagger}$

The chiral anomaly

$$\mathscr{L} = \mathrm{i}\,\Psi_{\mathrm{L}}^{\dagger}(\partial_t + \partial_x)\Psi_{\mathrm{L}} + \mathrm{i}\,\Psi_{\mathrm{R}}^{\dagger}(\partial_t - \partial_x)\Psi_{\mathrm{R}}$$

The chiral anomaly is an anomaly of $(\mathrm{U}(1)^V \times \mathrm{U}(1)^A)/\mathbb{Z}_2$

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Manifests through anomalous current conservation

$$\partial^{\mu} J_{\mu}^{A} = 0 \xrightarrow{\text{turn on } A_{\mu}} \partial^{\mu} J_{\mu}^{A} = \frac{1}{\pi} E$$

- The obstruction to a trivial gapped phase follows from

   1. Formally: 't Hooft's anomaly matching argument

   2. Physically: threading 2π flux creates $Q^A = 2$ charge—a left-moving particle and right-moving hole

The chiral anomaly

Can the chiral anomaly be realized in a lattice model with finite-dimensional local Hilbert spaces?

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- Not verbatim by a lattice $(\mathrm{U}(1)^{V}\times\mathrm{U}(1)^{A})/\mathbb{Z}_2$ symmetry

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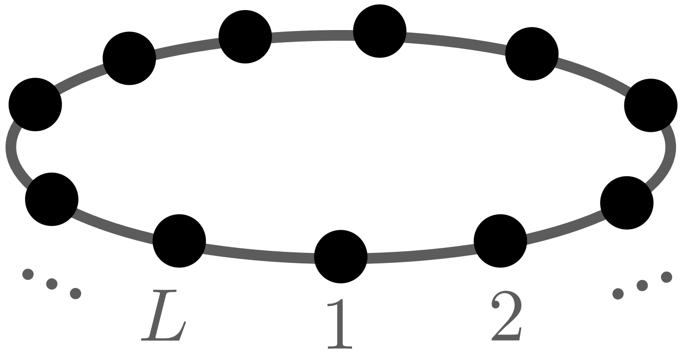
Yes!

A simple lattice model

Complex fermions c_j on sites j of a 1d spatial lattice*



$$\{c_j, c_{j'}^\dagger\} = \delta_{j,j'} \qquad \{c_j, c_{j'}\} = 0$$

$$H = \mathrm{i} \sum_{j=1}^{L} \left(c_j^\dagger c_{j+1} - c_{j+1}^\dagger c_j \right)$$

Becomes free, massless Dirac fermion theory in IR

---

* *Assume L is even and periodic boundary conditions*

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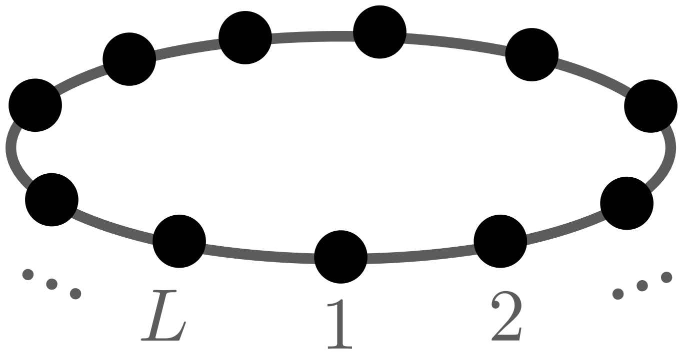

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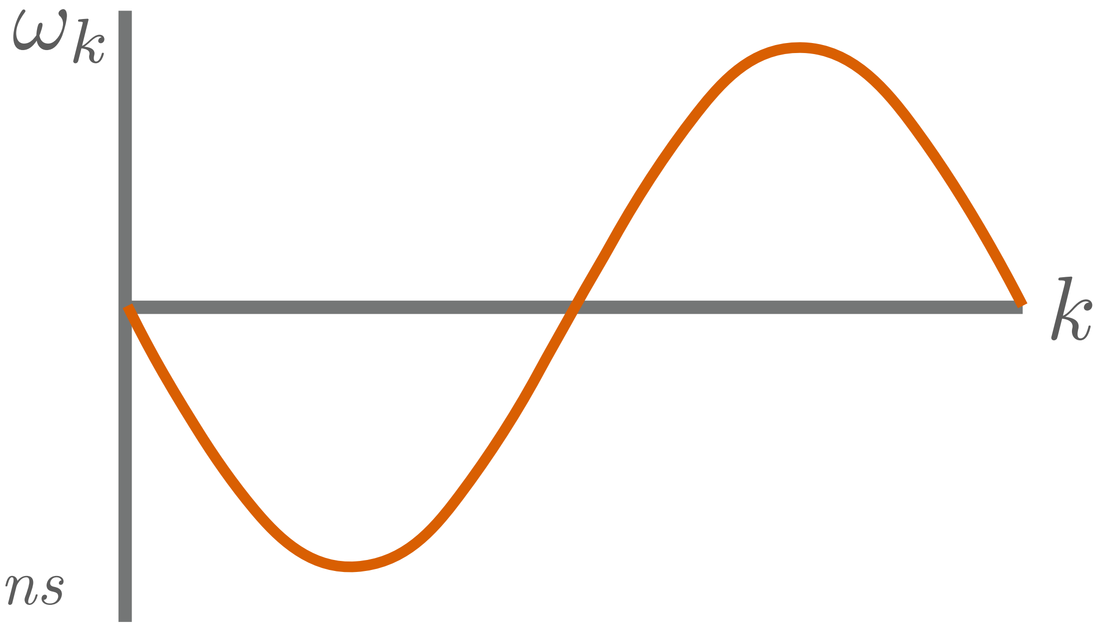
- In momentum space

$$H = \sum_{k \in \mathrm{BZ}} \omega_k c_k^\dagger c_k$$

$$\omega_k = -2 \sin(k)$$



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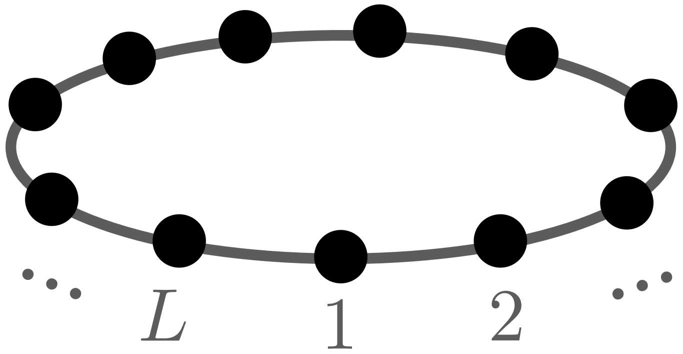

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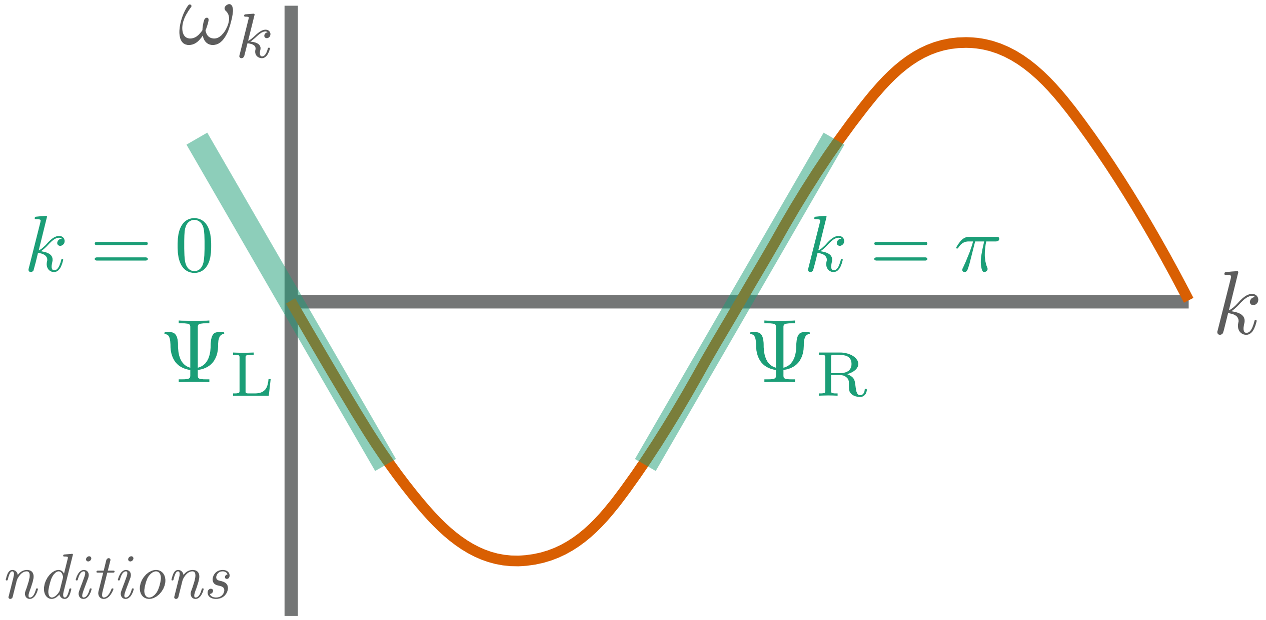

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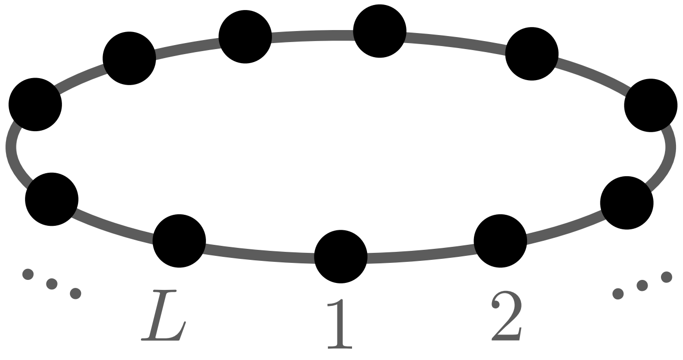

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Becomes free, massless Dirac fermion theory in IR

If the chiral anomaly emanates from a lattice anomaly, the chiral $(\mathrm{U}(1)^V \times \mathrm{U}(1)^A)/\mathbb{Z}_2$ symmetry must emanate from a lattice symmetry.

We need to build a UV to IR symmetry dictionary!

---

[*] *Assume L is even*

Emanant symmetries I

$$H = \mathrm{i}\sum_{j=1}^{L}\left(c_j^\dagger c_{j+1} - c_{j+1}^\dagger c_j\right) \qquad c_k = \frac{1}{\sqrt{L}}\sum_{j=1}^{L} \mathrm{e}^{\mathrm{i}kj} c_j$$

U(1) fermion number symmetry $N = \sum_{j=1}^{L} c_j^\dagger c_j$

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U(1) fermion number symmetry $Q_0 = \sum_{j=1}^{L}\left(c_j^\dagger c_j - \frac{1}{2}\right)$

- Real space transformation

$$\mathrm{e}^{\mathrm{i}\theta Q_0} : c_j^\dagger \mapsto \mathrm{e}^{\mathrm{i}\theta} c_j^\dagger$$

- Momentum space transformation

$$\mathrm{e}^{\mathrm{i}\theta Q_0} : c_k^\dagger \mapsto \mathrm{e}^{\mathrm{i}\theta} c_k^\dagger$$

- IR symmetry (look at $k = 0$ and $k = \pi$)

$$\mathrm{e}^{\mathrm{i}\theta Q_0} : \Psi_{L,R}^\dagger \mapsto \mathrm{e}^{\mathrm{i}\theta} \Psi_{L,R}^\dagger$$

Emanant symmetries I

Lattice symmetry	$\xrightarrow{\text{emanates to}}$	IR symmetry
$e^{i\theta Q_0}$		$e^{i\theta Q^V}$

$$e^{i\theta Q_0} : \Psi^\dagger_{L,R} \mapsto e^{i\theta} \Psi^\dagger_{L,R}$$

Emanant symmetries II

$$H = \mathrm{i}\sum_{j=1}^{L}\left(c_j^\dagger c_{j+1} - c_{j+1}^\dagger c_j\right) \qquad c_k = \frac{1}{\sqrt{L}}\sum_{j=1}^{L}\mathrm{e}^{\mathrm{i}kj}c_j$$

Lattice translation symmetry:

- Real space transformation

$$T\colon c_j \mapsto c_{j+1}$$

- Momentum space transformation

$$T\colon c_k \mapsto \mathrm{e}^{-\mathrm{i}k}c_k$$

- IR symmetry (look at $k=0$ and $k=\pi$)

$$T\colon \Psi_{\mathrm{L}}, \Psi_{\mathrm{R}} \mapsto \Psi_{\mathrm{L}}, -\Psi_{\mathrm{R}}$$

Emanant symmetries II

Lattice symmetry	$\xrightarrow{\text{emanates to}}$	IR symmetry
$\mathrm{e}^{\mathrm{i}\theta Q_0}$		$\mathrm{e}^{\mathrm{i}\theta Q^V}$
T		$\mathrm{e}^{\mathrm{i}\pi Q_R} = \mathrm{e}^{\frac{\mathrm{i}\pi}{2}(Q^V - Q^A)}$

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Emanant symmetries II

Lattice symmetry	$\xrightarrow{\text{emanates to}}$	IR symmetry
$\mathrm{e}^{\mathrm{i}\theta Q_0}$		$\mathrm{e}^{\mathrm{i}\theta Q^V}$
T		$\mathrm{e}^{\mathrm{i}\pi Q_R} = \mathrm{e}^{\frac{\mathrm{i}\pi}{2}(Q^V - Q^A)}$

$\mathrm{e}^{\mathrm{i}\pi Q_R}$ generates an anomalous chiral symmetry in the IR

- $\mathbb{Z}_2^R$ emanates from lattice translations
- But T is anomaly-free: this IR anomaly is emergent

$$T\colon \Psi_{\mathrm{L}}, \Psi_{\mathrm{R}} \mapsto \Psi_{\mathrm{L}}, -\Psi_{\mathrm{R}}$$

Be real 😎

Let's decompose c_j into real (Majorana) fermions $a_j = a_j^\dagger$ and $b_j = b_j^\dagger$ to search for other useful symmetries

$$c_j = \frac{1}{2}(a_j + \mathrm{i} b_j) \qquad \{a_j, a_{j'}\} = 2\delta_{j,j'} \qquad \{b_j, b_{j'}\} = 2\delta_{j,j'}$$

Hamiltonian becomes $H = \frac{\mathrm{i}}{2}\sum_{j=1}^{L}(a_j a_{j+1} + b_j b_{j+1})$

Be real 😎

Let's decompose c_j into real (Majorana) fermions $a_j = a_j^\dagger$ and $b_j = b_j^\dagger$ to search for other useful symmetries

$$c_j = \frac{1}{2}(a_j + \mathrm{i} b_j) \qquad \{a_j, a_{j'}\} = 2\delta_{j,j'} \qquad \{b_j, b_{j'}\} = 2\delta_{j,j'}$$

Hamiltonian becomes $H = \dfrac{\mathrm{i}}{2}\displaystyle\sum_{j=1}^{L}(a_j a_{j+1} + b_j b_{j+1})$

- The a_j and b_j Majoranas are decoupled!
- The model has Majorana translation symmetries

$$T_a : a_j, b_j \mapsto a_{j+1}, b_j$$

$$T_b : a_j, b_j \mapsto a_j, b_{j+1}$$

Emanant symmetries III

$$H = \frac{\mathrm{i}}{2}\sum_{j=1}^{L}(a_j a_{j+1} + b_j b_{j+1}) \qquad c_j = \frac{1}{2}(a_j + \mathrm{i} b_j)$$

The b Majorana lattice translation symmetry

- Real space transformation:

$$T_b : c_j \mapsto \frac{1}{2}\left(a_j + \mathrm{i} b_{j+1}\right)$$

- Momentum space transformation

$$T_b : c_k = \frac{1}{2}\left(a_k + \mathrm{i} b_k\right) \mapsto \frac{1}{2}\left(a_k + \mathrm{e}^{-\mathrm{i}k} \mathrm{i} b_k\right)$$

- IR symmetry (look at $k = 0$ and $k = \pi$)

$$T_b : \Psi_{\mathrm{L}}, \Psi_{\mathrm{R}} \mapsto \Psi_{\mathrm{L}}, \Psi_{\mathrm{R}}^{\dagger}$$

Emanant symmetries III

Lattice symmetry	$\xrightarrow{\text{emanates to}}$	IR symmetry
$e^{i\theta Q_0}$		$e^{i\theta Q^V}$
T		$e^{i\pi Q_R} = e^{\frac{i\pi}{2}(Q^V - Q^A)}$
T_b		C_R

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$\mathrm{e}^{\mathrm{i}\alpha\,(T_b Q_0 T_b^\dagger)}$		$\mathrm{e}^{\mathrm{i}\alpha\,(C_R Q^V C_R^\dagger)} = \mathrm{e}^{\mathrm{i}\alpha Q^A}$

$$T_b : \Psi_{\mathrm{L}}, \Psi_{\mathrm{R}} \mapsto \Psi_{\mathrm{L}}, \Psi_{\mathrm{R}}^\dagger$$

Lattice vector and axial charges

The IR vector and axial charges emanate from the conserved charges

- Lattice vector charge $Q_0 = \frac{\mathrm{i}}{2}\sum_{j=1}^{L} a_j b_j$

- Lattice axial charge $Q_1 \equiv T_b Q_0 T_b^{-1} = \frac{\mathrm{i}}{2}\sum_{j=1}^{L} a_j b_{j+1}$

  1. Sum of local terms
  2. Have integer-quantized eigenvalues
  3. Generate locality preserving U(1) symmetries

Onsager symmetry

The lattice vector and axial charges do not commute

$$[Q_0, Q_1] \neq 0$$

Onsager symmetry

The lattice vector and axial charges do not commute

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- Generate the Onsager algebra [Onsager '44]

$$[Q_n, Q_m] = \mathrm{i} G_{m-n} \qquad [G_n, G_m] = 0$$

$$[Q_n, G_m] = 2\mathrm{i}(Q_{n-m} - Q_{n+m})$$

with $Q_n = \frac{\mathrm{i}}{2} \sum_{j=1}^{L} a_j b_{j+n}$ and $G_n = \frac{\mathrm{i}}{2} \sum_{j=1}^{L} (a_j a_{j+n} - b_j b_{j+n})$

Onsager symmetry

The chiral $(\mathrm{U}(1)^V \times \mathrm{U}(1)^A)/\mathbb{Z}_2$ symmetry emanates from the Onsager symmetry $\langle \mathrm{e}^{\mathrm{i}\theta Q_0}, \mathrm{e}^{\mathrm{i}\alpha Q_1} \rangle$

$$Q_n \xrightarrow{\text{IR limit}} \begin{cases} Q^V & n \text{ even} \\ Q^A & n \text{ odd} \end{cases} \qquad G_n \xrightarrow{\text{IR limit}} 0$$

$$[Q_0, Q_1] \neq 0 \xrightarrow{\text{IR limit}} [Q^V, Q^A] = 0$$

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Does the Onsager symmetry have a lattice anomaly that matches the chiral anomaly?

Onsager symmetric Hamiltonians

We assume the Hamiltonian is local:

$$H_g = \sum_n \sum_{j=1}^{L} g_{j,n} H_j^{(n)}$$

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1. $e^{-\mathrm{i}\frac{\pi}{2}Q_1} e^{\mathrm{i}\frac{\pi}{2}Q_0} : (a_j, b_j) \mapsto (a_{j-1}, b_{j+1})$ invariance requires $H_j^{(n)}$ to not have terms mixing a_j and b_j and $g_{j,n} = g_n$

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2. Under the $\mathrm{e}^{\mathrm{i}\theta Q_0}$ transformation

$$a_j \to \cos(\theta) a_j + \sin(\theta) b_j \qquad b_j \to \cos(\theta) b_j - \sin(\theta) a_j$$

$\Longrightarrow$ Symmetric $H_j^{(n)}$ are quadratic

$$H_j^{(n)} = \mathrm{i} a_j a_{j+n} + \mathrm{i} b_j b_{j+n}$$

Lattice anomaly: enforced gaplessness

$$H_g = \mathrm{i} \sum_n \sum_{j=1}^{L} g_n \left(a_j a_{j+n} + b_j b_{j+n}\right)$$

- H_g commutes with the entire Onsager symmetry — it is the most general Onsager symmetric Hamiltonian

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Every Onsager symmetric Hamiltonian H_g is gapless

- In momentum space:

$$H_g = \sum_{k \in \mathrm{BZ}} \omega_k c_k^\dagger c_k \qquad \omega_k = 4 \sum_n g_n \sin(nk)$$

- H_g is never in a trivial gapped phase
- This Onsager symmetry has a lattice anomaly

Lattice anomaly: enforced gaplessness

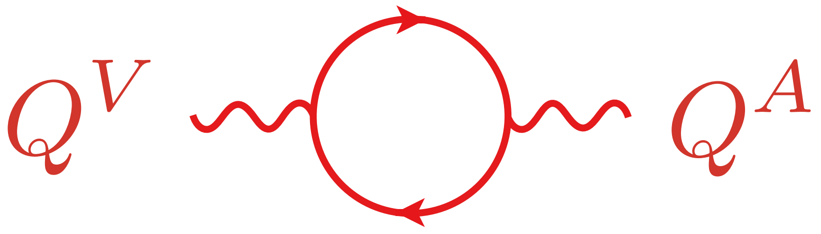
$$H_g = \mathrm{i}\sum\sum^{L} g_n\left(a_j a_{j+n} + b_j b_{j+n}\right)$$

- ➤

Ev

The chiral anomaly emanates from the Onsager symmetry's lattice anomaly

$$[Q_0\,, Q_1\,] \neq 0 \xrightarrow{\text{RG}}$$



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Recap

Anomalies 101

1. Anomalies as obstructions to trivial gapped phases
2. Emergent vs emanant anomalies

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Anomalies 101

1. Anomalies as obstructions to trivial gapped phases
2. Emergent vs emanant anomalies

Lattice chiral anomaly

[Arkya Chatterjee, **Sal Pace**, Shu-Heng Shao, PRL '25 (arXiv:2409.12220)]

The simple tight-binding model $H = \mathrm{i} \sum_{j=1}^{L} \left(c_j^\dagger c_{j+1} - c_{j+1}^\dagger c_j \right)$

1. Has a lattice vector and axial symmetry, which form the Onsager algebra
2. Has a lattice anomaly that enforces gaplessness, from which the chiral anomaly emanates

Recap

Anomalies 101

Other lattice anomalies from Onsager-like symmetries:

- Compact boson CFT anomalies in spin chains
  [**Sal Pace**, Arkya Chatterjee, Shu-Heng Shao, SciPost Phys '25 (arXiv:2412.18606)]
- Witten's SU(2) anomaly on the lattice
  [L. Gioia, R. Thorngren, PRL '26 (arXiv:2503.07708)]
- Lattice parity anomaly: symmetry-enforced Dirac cones
  [**Sal Pace**, Luke Kim, Arkya Chatterjee, Shu-Heng Shao PRL '25 (arXiv:2505.04684)]
- Symmetry-enforced Fermi surfaces
  [Luke Kim, **Sal Pace**, Shu-Heng Shao, PRL '26 (arXiv:2512.04150)]

which the chiral anomaly emanates

Back-up slides

Lattice vector and axial charges

In terms of complex fermions

- Lattice vector charge $Q_0 = \sum_{j=1}^{L} \left(c_j^\dagger c_j - \frac{1}{2} \right)$
- Lattice axial charge

$$Q_1 = \frac{1}{2} \sum_{j=1}^{L} \left(c_j^\dagger c_{j+1} - c_j c_{j+1}^\dagger + c_j c_{j+1} - c_j^\dagger c_{j+1}^\dagger \right)$$

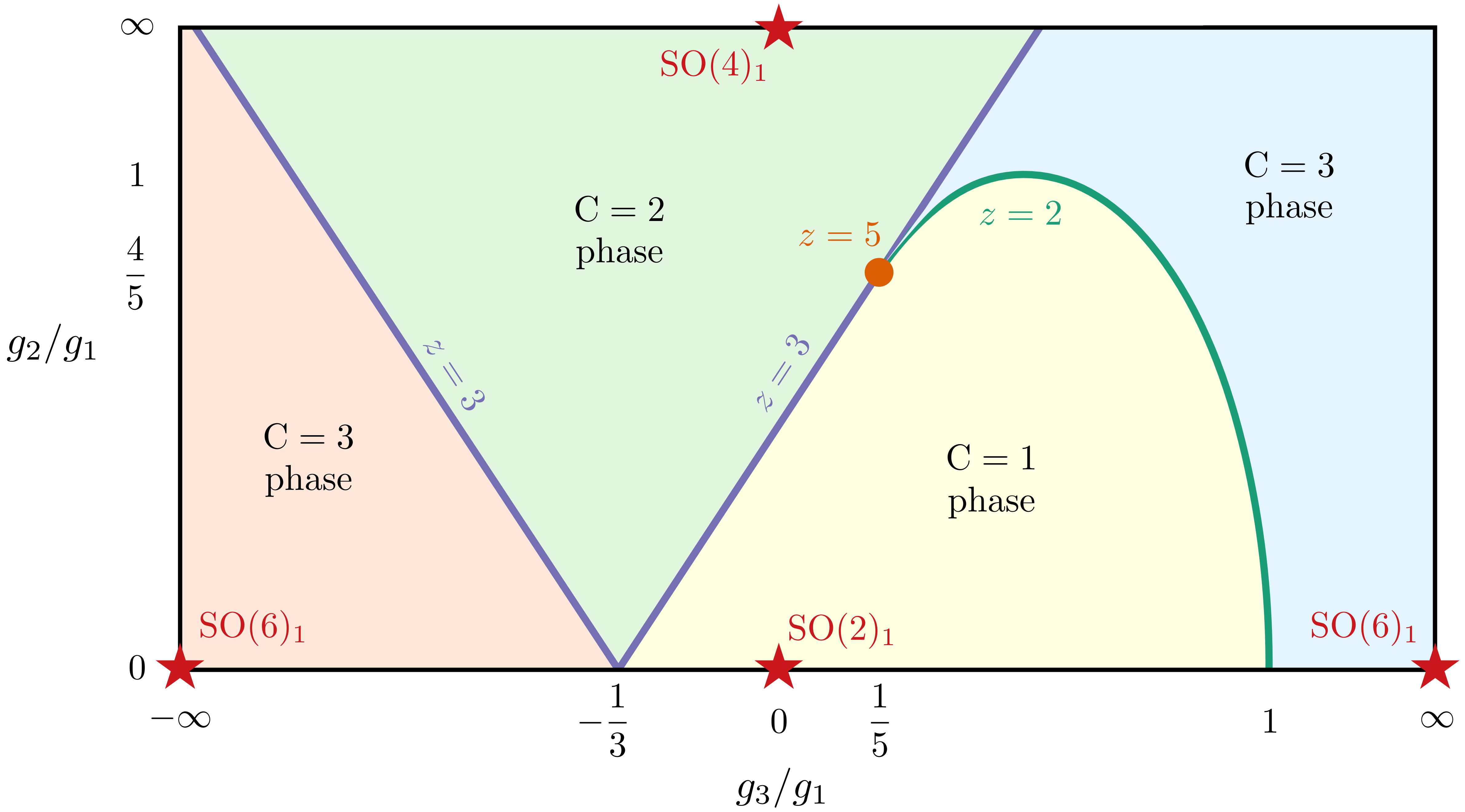
- Generate locality preserving U(1) symmetries

$$\mathrm{e}^{\mathrm{i}\theta Q_0} : c_j \mapsto \mathrm{e}^{-\mathrm{i}\theta} c_j$$

$$\mathrm{e}^{\mathrm{i}\alpha Q_1} : c_j \mapsto \cos(\alpha) c_j - \frac{\mathrm{i}}{2} \sin(\alpha) (c_{j-1}^\dagger + c_{j-1} - c_{j+1}^\dagger + c_{j+1})$$

Gapless phase diagram

$$H = \mathrm{i}\sum_{n=1}^{3}\sum_{j=1}^{L} g_n\left(a_j a_{j+n} + b_j b_{j+n}\right)$$



An anomaly in quantum spin chains

Consider model with a qubit on each site j of a length L ring

$$\mathscr{H} = \bigotimes_{j=1}^{L} \mathbb{C}^2 \qquad X_j = X_{j+L} \qquad Z_j = Z_{j+L}$$

Anomalous $\mathbb{Z}_2^X \times \mathbb{Z}_2^Z \times$(lattice translations) symmetry

$$U_X = \prod_{j=1}^{L} X_j \qquad U_Z = \prod_{j=1}^{L} Z_j \qquad T : j \mapsto j+1$$

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Manifestations of the anomaly:

- U_X and U_Z have a QM anomaly in each unit cell
- $U_X U_Z = (-1)^L U_Z U_X$

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Every symmetric Hamiltonian cannot have a trivial gapped phase

- e.g., XX chain $H = \sum_{j=1}^{L} (X_j X_{j+1} + Y_j Y_{j+1})$
- Called a Lieb-Schultz-Mattis (LSM) anomaly

An anomaly in field theory

The compact boson CFT at radius R is a $1+1$D CFT with

$$\mathscr{L}_R = \frac{R^2}{4\pi}\partial_\mu \Phi \partial^\mu \Phi \qquad \Phi \sim \Phi + 2\pi$$

- Has a $(\mathrm{U}(1)^M \times \mathrm{U}(1)^W) \rtimes \mathbb{Z}_2^C$ symmetry:

$$J_\mu^M = \frac{R^2}{2\pi}\partial_\mu \Phi \qquad J_\mu^W = \frac{1}{2\pi}\epsilon_{\mu\nu}\partial^\nu \Phi \qquad C : \Phi \mapsto -\Phi$$

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Anomalous symmetries:

- $\mathrm{U}(1)^M \times \mathrm{U}(1)^W$ and $\mathbb{Z}_2^M \times \mathbb{Z}_2^W \times \mathbb{Z}_2^C$
- Manifestation of $\mathrm{U}(1)^M \times \mathrm{U}(1)^W$ anomaly:

$$\partial^\mu J_\mu^W = 0 \xrightarrow{\text{turn on } A_\mu^M} \partial^\mu J_\mu^W = \frac{1}{2\pi}E^M$$

Anomaly matching

Anomalies of the effective IR theory either emerge from nothing or emanate from an anomaly of the UV

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Anomalies of the effective IR theory either emerge from nothing or emanate from an anomaly of the UV

Example: XX spin chain → $R = \sqrt{2}$ boson CFT

$\mathbb{Z}_2^X$	$\mathbb{Z}_2^C$
$\mathbb{Z}_2^Z$	$\mathbb{Z}_2^M$
Translations	$\mathbb{Z}_2^{\text{diag}} \subset \mathbb{Z}_2^M \times \mathbb{Z}_2^W$

[M. A. Metlitski, Thorngren '17; M. Cheng, N. Seiberg '22]

- ➤ Anomaly of $\mathbb{Z}_2^{\text{diag}}$ is emergent
- ➤ Anomaly of $\mathbb{Z}_2^M \times \mathbb{Z}_2^W \times \mathbb{Z}_2^C$ emanates from LSM anomaly