

Symmetry in Quantum Lattice Systems.

Lecture plan

1) Unitary Symmetry operators on the lattice

Nachtergaele, Sims, Young '18 ; Freedman, Haah, Hastings '19 ; Kopustin, Sopenko '24

2) Symmetry defects and anomalies on the lattice

Else, Nagak '14 ; Cheng, Seiberg '22 ; Seifnashri '23 ; Kopustin, Sopenko '24

3) Generalized Symmetry on the lattice,

Qi, Rudzikhovskiy, Hermele '20 ; Seiberg, Shao '23 ; Okada, Tachikawa '24 ;
SP, Lam, AKroy '25

Goal: Introduce basic aspects of the framework with examples
(less emphasis on applications due to time.)

Why care about the lattice?

1) QFT pov: Mathematically controlled setting for non-perturbative physics.
Computational study and quantum-simulation

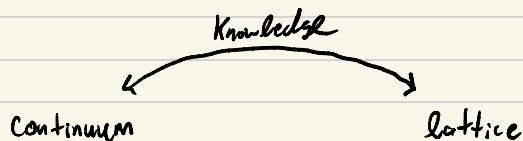
2) Quantum lattice matter: Condensed matter and AMO systems.

Two slogans:

1) Sometimes the lattice is a regulator. Sometimes it's the physics.

2) Sometimes the QFT is the starting point. Sometimes it's the answer.

Mutualistic relationship



→ Modern examples

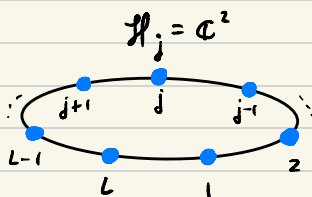
Foliated field theory ← Fractons

Generalized Symmetry → New lattice models
with interesting phases and phase transitions

Hamiltonian lattice system: Spatial lattice and continuous time.

Example: Ising Chain (Warm up example)

1+1d lattice system on a ring with a qubit on each site j and total # of sites = L → aka spin 1/2.



Onsite tensor product Hilbert Space

$$\mathcal{H} = \bigotimes_{j=1}^L \mathbb{C}^2$$

→ site j Pauli operators X_j and Z_j ($j \sim j+L$):

$$X_j^2 = Z_j^2 = 1 \quad X_j Z_k = (-1)^{\delta_{j,k}} Z_k X_j$$

$$\text{Hamiltonian } H = - \sum_{j=1}^L Z_j Z_{j+1} - h \sum_{j=1}^L X_j \quad (h \in \mathbb{R})$$

→ Locality: Each term involves at most neighboring qubits

General Set up

A $(n+1)d$ lattice system is Specified by:

→ or other dimensional cells

- 1) Spatial lattice: Discrete Set of lattice sites Λ with an n -dimensional metric Space X and locally finite, injective map

$$\Lambda \rightarrow X$$

→ X gives a notion of spatial dimension and distance

- 2) Onsite Hilbert spaces $\{\mathcal{H}_j\}_{j \in \Lambda}$

Hilbert space in finite region Γ : $\mathcal{H}_\Gamma = \bigotimes_{j \in \Gamma} \mathcal{H}_j$.

→ For finite Λ , total Hilbert space is $\mathcal{H}_\Lambda$.

→ For infinite Λ , $\mathcal{H}_\Lambda$ is ill-defined.

e.g., For $|\psi\rangle = \bigotimes_{j \in \Lambda} |\psi_j\rangle$ and $|\phi\rangle = \bigotimes_{j \in \Lambda} |\phi_j\rangle$, the inner-product $\langle \psi | \phi \rangle = \prod_{j \in \Lambda} \langle \psi_j | \phi_j \rangle$ doesn't converge.

(Total Hilbert space found using a reference state: GNS construction)

- 3) Local projectors $\{P_j\}_{j \in \Lambda}$

→ A local operator O is physical iff $O P_j = P_j O$

- 4) Hamiltonian $H = \sum_{r \in \Lambda} h_r \otimes \mathbb{1}_{\Lambda \setminus r}$

→ Each h_r a Hermitian operator of $\mathcal{H}_r$.

→ Strict-locality: $h_r = 0$ if $\text{diam}(\Gamma) > \Gamma_H$ with Γ_H the finite range of H .

→ Independent of $|\Lambda|$ for finite Λ .

→ For infinite Λ , only $[O, H] = i\partial_t O$ is a well defined operator.

$$[O, H] = \sum_{n: n \cap \text{Supp}(O) \neq \emptyset} [O, h_n \otimes \mathbb{1}_{\Lambda \setminus n}] \quad \left. \vphantom{\sum} \right\} \text{A finite sum.}$$

Comments:

1) locality baked into the Set up.

2) Infinite Λ is subtle (we'll try to stick to finite Λ)

We will assume: (For simplicity)

1) $\dim \mathcal{H}_i < \infty$

2) Lectures 1 & 2: No nontrivial projectors P_j .

3) Lectures 1 & 2 in 1+1d, Lecture 3 in 1+1d and 2+1d

Grouplike Symmetry

An operator U is a symmetry operator iff U is

1) (anti-)Unitary: $U^\dagger = U^{-1}$ (we'll restrict to unitaries)

Not needed in 0+1d QM. $\left\{ \begin{array}{l} 2) \text{locality-respecting: } U \circ U^\dagger \text{ is a local operator for each local operator } O \end{array} \right.$

→ a group
A G -Symmetry is a collection $\{U_g\}_{g \in G}$ of symmetry operators satisfying

1) Group law: $U_g U_h = e^{i\Theta(g,h)} U_{gh}$.

2) Faithfulness: $U_g \propto \mathbb{1}$ iff $g=1$.

The Symmetry of a Hamiltonian H is a G -Symmetry $\{U_g\}_{g \in G}$ such that

$$[U_g, H] = 0 \quad \forall g$$

→ Note: Symmetry can be stated without a reference symmetric Hamiltonian in mind.

Ising Chain Symmetry:

Consider the operators

$\eta = \prod_{j=1}^L X_j$	$\eta X_j \eta^\dagger = X_j$	$\eta Z_j \eta^\dagger = -Z_j$
translation operator T	$T X_j T^\dagger = X_{j+1}$	$T Z_j T^\dagger = Z_{j+1}$
reflection operator R	$R X_j R^\dagger = X_{-j}$	$R Z_j R^\dagger = Z_{-j}$

→ All are Symmetry operators

→ Satisfy

$$\eta^2 = 1$$

$$T^L = 1$$

$$R^2 = 1$$

$$\eta T = T \eta$$

$$\eta R = R \eta$$

$$R T = T^{-1} R$$

⇒ Generate a $\mathbb{Z}_2 \times D_{2L}$ Symmetry

→ Satisfy $[\eta, H] = [T, H] = [R, H] = 0$

⇒ $\mathbb{Z}_2 \times D_{2L}$ Symmetry of the Ising Chain.

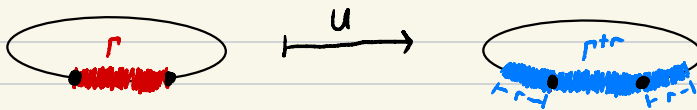
Quantum Cellular Automaton (QCA)

A special class of Unitary Symmetry operators are QCAs.

→ U is a QCA if for each local operator o supported in Γ ,

$$\text{Supp}(U o U^\dagger) \subseteq \Gamma^{+r}$$

With Γ^{+r} all sites within a finite distance r of Γ .



⇒ Minimal such r is called the range of U .

→ QCAs generate strictly locality-preserving symmetries.

In the Ising chain:

Symmetry operator	QCA?
$\gamma = \prod_{j=1}^L X_j$	Yes (range 0)
translation T	Yes (range 1)
reflection R	No

A QCA U with range 0 is called an onsite Unitary

→ e.g., γ is onsite.

$\rightarrow U$ is onsite $\Leftrightarrow U = \bigotimes_{j \in \Lambda} U_j$ with $U_j \in \mathcal{U}(\mathcal{H}_j)$

Finite-depth quantum circuit (FDQC)

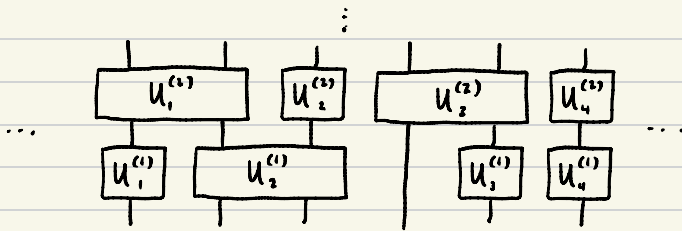
A QCA U is a FDQC with depth D iff $\rightarrow$ finite in thermodynamic limit.

$$U = U^{(D)} U^{(D-1)} \dots U^{(1)}, \quad U^{(k)} = \prod_a U_a^{(k)}$$

with $\{U_a^{(k)}\}$ local unitary operators satisfying

$$\text{Supp}(U_a^{(k)}) \cap \text{Supp}(U_b^{(k)}) = \emptyset \quad \forall a \neq b.$$

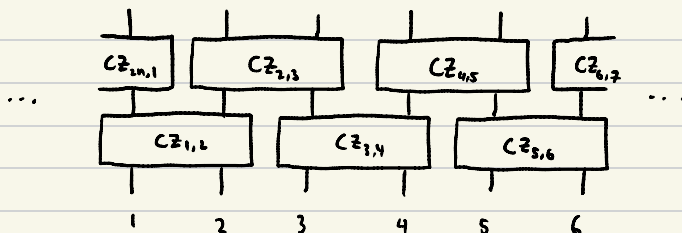
$\rightarrow U_a^{(k)}$ are called gates.



$$\{\text{Onsite Unitaries}\} \subset \{\text{FDQCs}\} \subset \{\text{QCAs}\}$$

with $CZ_{j,k} = \frac{1}{2}(1 + z_j + z_k - z_j z_k)$

Example: $\prod_{j=1}^{2n} CZ_{j,j+1} = \prod_{\text{even } j} CZ_{j,j+1} \prod_{\text{odd } j} CZ_{j,j+1}$



Brick-wall circuit.

Anti-example: Lattice translation T

[Gross - Neeme - Vogts - Werner (GNVW) 2009]

GNVW index

Every QCA U has an index $\text{ind}(U)$

→ Measures net transport quantum information across a spatial cut.

$$\Rightarrow \text{For translation } T O_j T^\dagger = O_{j+1}$$

$$\text{ind}(T) = \dim \mathcal{H}_j.$$

For bosonic 1+1d lattice systems

(Won't give full def here)

$$\text{ind}(U) = \text{GNVW index of } U \in \mathcal{Q}_{\geq 0}$$

Group homomorphism $\text{ind}: G_{\text{QCA}} \rightarrow \mathcal{Q}_{\geq 0}$

$$\text{ind}(UV) = \text{ind}(U)\text{ind}(V) \quad \text{ind}(U^{-1}) = \text{ind}(U)^{-1}$$

→ Satisfies

$$\text{ind}(U) = 1 \iff U \text{ is an FDQC} \quad \text{ind}(U \otimes V) = \text{ind}(U)\text{ind}(V)$$

⇒ T is not an FDQC

Examples:

$$1) \text{ qubit translations: } \text{ind}(T_{\text{qubit}}) = 2, \quad \text{ind}(T_{\text{qubit}}^{-1}) = 1/2$$

$$2) \text{ qutrit translations: } \text{ind}(T_{\text{qutrit}}) = 3, \quad \text{ind}(T_{\text{qutrit}}^{-1}) = 1/3$$

$$3) \mathcal{H}_j = \mathbb{C}^2 \otimes \mathbb{C}^3: \quad \text{ind}(T_{\text{qubit}} T_{\text{qutrit}}) = 6, \quad \text{ind}(T_{\text{qubit}} T_{\text{qutrit}}^{-1}) = 2/3$$

4) For prime $\dim \mathcal{H}_j = p$ Every QCA U admits decomposition $U = T^n C$ with C an FDQC

$$\text{ind}(U) = \text{ind}(T)^n \text{ind}(C) = p^n.$$

Remarks:

1) For every QCA U , $U \otimes U^{-1}$ is an FDQC

$$\rightarrow \text{ind}(U \otimes U^{-1}) = \text{ind}(U) \text{ind}(U)^{-1} = 1$$

$\rightarrow$ Explicit expression:

let $U_A = U \otimes \mathbb{1}$, $U_B = \mathbb{1} \otimes U$, and $S = \prod_j S_j$ swaps the two tensor systems.

$$\Rightarrow U \otimes U^{-1} = U_A U_B^{-1} = U_A S U_A^{-1} S = \prod_j U_A S_j U_A^{-1} \prod_j S_j$$

$\rightarrow$ in thermodynamic limit

2) For a finite G -symmetry of QCA, $\{U_g\}_{g \in G}$, each U_g is an FDQC.

$\rightarrow$ Every U_g satisfies $U_g^{\text{ord}(g)} \propto \mathbb{1}$.

$$\Rightarrow \text{ind}(U_g^{\text{ord}(g)}) = 1$$

$$\Rightarrow \text{ind}(U_g) = 1$$

Lecture 2

Last lecture

- Lattice Hamiltonian Systems
- Unitary Symmetry operators: onsite $\subset$ FQCS $\subset$ QCA's

This lecture: defects and anomalies

Defects

A defect is a localized modification to a theory.

In a Hamiltonian theory, a defect D with support Σ modifies: ↗ in space

1) The Hilbert Space

$$\mathcal{H}_r = \bigotimes_{i \in r} \mathcal{H}_i \longmapsto \mathcal{H}_r^{(D, \Sigma)}$$
$$\rightarrow \mathcal{H}_r^{(D, \Sigma)} = \mathcal{H}_r \quad \text{if} \quad r \cap \Sigma = \emptyset.$$

↗ called defect Hilbert Space

2) Hamiltonian

$$H \longmapsto H_0^{(\Sigma)}$$

↗ called defect Hamiltonian.

$\rightarrow H_0^{(\Sigma)}$ acts like H away from Σ .

$\rightarrow$ If every $\mathcal{H}_r^{(D, \Sigma)} = \mathcal{H}_r$:

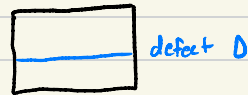
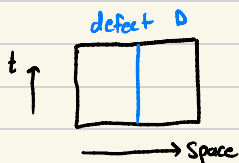
$$H_0^{(\Sigma)} = H + \delta H_0^{(\Sigma)}$$

with $\text{Supp}(\delta H_0^{(\Sigma)}) \subseteq \Sigma^{+w}$

$\rightarrow$ other operators also modified within Σ^{+w}

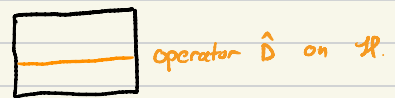
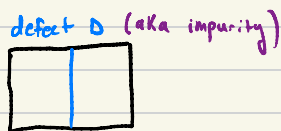
Spacetime pov comment

Euclidean theory: (pov of Iñaki's lecture)



→ Both are localized modifications of Euclidean path integral

Hamiltonian interpretation



Partition function:

$$\text{Tr}_{\mathcal{H}^{(0,z)}} [e^{-\beta H_0^{(z)}}]$$

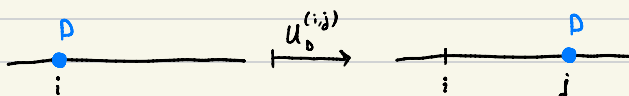
$$\text{Tr}_{\mathcal{H}} [\hat{D} e^{-\beta H}]$$

Symmetry defects

A symmetry defect is

1) a topological defect: a defect whose support is changed using unitary "movement" operators

e.g., 1d (picture in space, not spacetime)



Unitary $U_0^{(i,j)}$ has support $\text{Supp}(U_0^{(i,j)}) \subset [i, j]^{+w}$,

$$\Rightarrow H_0^{(j)} = U_0^{(i,j)} H_0^{(i)} U_0^{(i,j)+}$$

$\Rightarrow$ Energy spectrum independent of Symmetry defect's location

2) Implements a Symmetry transformation across space:

$$U_D^{(i,j)} O_K U_D^{(i,j)\dagger} = \begin{cases} \hat{D} O_K \hat{D}^\dagger & i \ll K \ll j \\ O_K & K \ll i \text{ or } K \gg j \end{cases}$$

$\Rightarrow D$ is the Symmetry defect for the Symmetry operator $\hat{D}$.

$$\Rightarrow \hat{D} = \prod_{j=1}^L U_D^{(j,j+1)} \quad (\text{defect} \rightarrow \text{op})$$

op $\rightarrow$ defect { For an FDQC U , a U -Symmetry defect can be inserted using a locality-preserving unitary U_B satisfying within B .

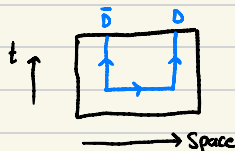
$$U_B O U_B^\dagger = \begin{cases} U O U^\dagger & \text{Supp}(O) \text{ deep inside of } B \\ 0 & \text{Supp}(O) \text{ far outside of } B \end{cases}$$

(U_B is a truncation of U to B)

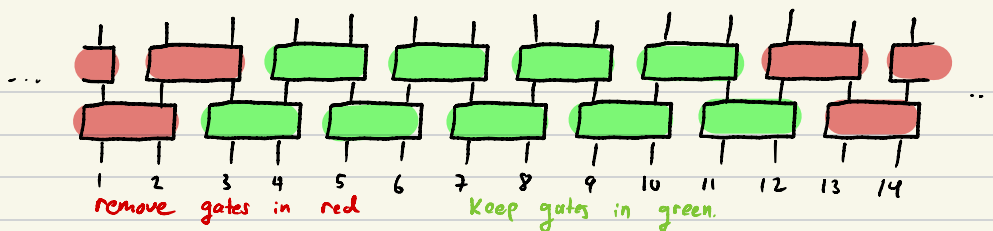
$\rightarrow U_B$ inserts a U -Symmetry defect at ∂B .

In 1+1d

$\rightarrow$ Spacetime intuition: truncated operator corresponds to Euclidean theory defect



$\rightarrow$ Example with depth 2 FDQC.



$\Rightarrow$ Truncation of the FDQC to the interval $[3, 12]$

$\rightarrow$ Fails for a QCA that is not an FDQC: A nontrivial GNVW index is an obstruction to a locality-preserving, unitary truncation.

(Because such QCAs transport information across space.)

Ising chain example

Ising chain with a η -Symmetry defect

1) Truncate η to interval $I = [j_L, j_R]$

$$\eta_I = \prod_{j=j_L}^{j_R} X_j$$

2) Conjugate H :

$$\eta_I H \eta_I^\dagger = H + 2 Z_{j_R-1} Z_{j_R} + 2 Z_{j_R} Z_{j_R+1}$$

3) Defect Hamiltonian: $H_\gamma^{(j_R)} = H + 2 Z_{j_R} Z_{j_R+1}$

$\rightarrow$ Movement operator $M_{j_R} = \eta_{[j_L, j_R+1]} \eta_I^\dagger = X_{j_R+1}$

$$\Rightarrow M_{j_R} H_\gamma^{(j_R)} M_{j_R}^\dagger = H_\gamma^{(j_R+1)}$$

Twisted translations:

$$\gamma_{\pm} T \gamma_{\pm}^{\dagger} = X_{j_L} X_{j_{R+1}} T \Rightarrow T_{\gamma}^{(in)} = X_{j_{R+1}} T$$

$$\rightarrow \text{Satisfies } (T_{\gamma}^{(in)})^L = \gamma.$$

Anomalies

Consider discrete G -symmetry $\{U_g\}_{g \in G}$ in 1+1d with

1) control space $X = S^1$

2) every U_g is a FDQC $\rightarrow$ automatically true for finite G with GCA symmetry operators

Truncate each U_g to U_g^I with $I = [j_L, j_R]$ and sites j_L and j_R far away.

$$U_g^I U_h^I = \Omega_{g,h} U_{gh}^I.$$

$\rightarrow$ An operator.

$\rightarrow$ Separation of j_L and j_R

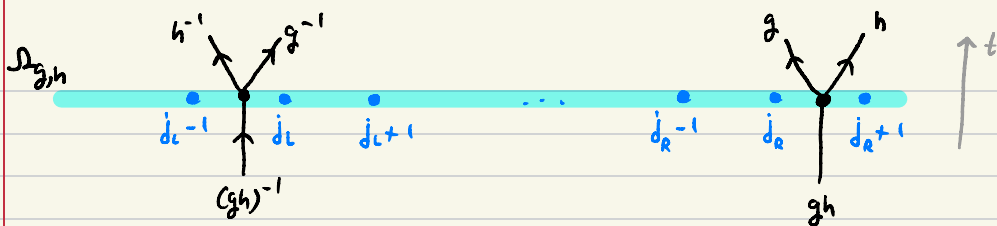
$$\Omega_{g,h} = \Omega_{g,h}^L \Omega_{g,h}^R$$

with $\Omega_{g,h}^{L/R}$ localized around $j_{L/R}$.

$\rightarrow$ assume $U_1^I = \mathbb{1} \Rightarrow \Omega_{1,g} = \Omega_{g,1} = \mathbb{1}$.

U_g^I creates a g -defect near j_R and g^{-1} defect near j_L .

$$\rightarrow \Omega_{g,h} = U_g^I U_h^I (U_{gh}^I)^{\dagger}:$$



$\Rightarrow \Omega_{g,h}^R$ is a "splitting" operator for a gh defect.

Associativity: $(U_g^I U_h^I) U_k^I = U_g^I (U_h^I U_k^I)$

$\Rightarrow \Omega_{g,h} \Omega_{gh,k} = (U_g^I \Omega_{h,k} (U_g^I)^+) \Omega_{g,hk}$

$\Rightarrow \Omega_{g,h}^R \Omega_{gh,k}^R = \omega(g,h,k) (U_g^I \Omega_{h,k}^R (U_g^I)^+) \Omega_{g,hk}^R$

Where $\omega: G^3 \rightarrow U(1)$

Graphically

The anomaly of the G -symmetry is classified by the cohomology class

$$[\omega] \in \mathcal{H}^3(G, U(1))$$

$\rightarrow$ From its def., ω satisfies the 3-cocycle condition

→ Redefining $\Omega_{g,h}^R \mapsto e^{i\theta(g,h)} \Omega_{g,h}^R$ changes ω by a 3-coboundary

Note: Agrees with the classification of 't Hooft anomalies in QFT.

(Suprising: The lattice is non-relativistic. No continuum limit needed)

Example: CZX Symmetry

Consider 1d lattice system of even L sites with a qubit per site, and the FQCC

$$U = \prod_{j=1}^L CZ_{j,j+1} \prod_{j=1}^L X_j$$

→ Controlled-Z gate $CZ_{j,k} = \frac{1}{2}(1 + Z_j + Z_k - Z_j Z_k)$

$$(CZ_{j,k} X_j CZ_{j,k}^\dagger = X_j Z_k, \quad CZ_{j,k} X_k CZ_{j,k}^\dagger = X_k Z_j)$$

U satisfies:

$$U X_j U^\dagger = Z_{j-1} X_j Z_{j+1}, \quad U Z_j U^\dagger = -Z_j$$

$\Rightarrow U^2 = 1 \Rightarrow U$ generates a $\mathbb{Z}_2$ symmetry

A symmetric Hamiltonian:

$$H = \frac{1}{2} \sum_{j=1}^L (X_j + Z_{j-1} X_j Z_{j+1})$$

Fun facts: 1) IR limit is $R=\sqrt{2}$ compact boson CFT

2) critical point between inequivalent $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPT phases

U-Symmetry defect

1) Truncate U to interval $I = [j_L, j_R]$

$$U_I = \prod_{j=j_L}^{j_R-1} C Z_{j, j+1} \prod_{j=j_L}^{j_R} X_j$$

2) Conjugate H :

$$U_I H U_I^\dagger = H + \delta H_L + \delta H_R$$

with, e.g.,

$$\delta H_R = -X_{j_R} Z_{j_R, j_R+1} C Z_{j_R-1, j_R+1} - Z_{j_R} X_{j_R+1} Z_{j_R+2}$$

3) Defect Hamiltonian: $H_U^{(j_R)} = H + \delta H_R$.

→ Movement operator

$$M_{j_R} = U_{[j_L, j_R+1]} U_I^\dagger = C Z_{j_R, j_R+1} X_{j_R+1}$$

Anomaly of $\mathbb{Z}_2$ Symmetry $\{U^n\}_{n \in \mathbb{Z}_2}$.

$$U_I^2 = (-1)^{|I|-1} Z_{j_L} Z_{j_R} \Rightarrow \Omega_{m,n} = \left((-1)^{|I|-1} Z_{j_L} Z_{j_R} \right)^{mn}$$

$$\Rightarrow \Omega_{m,n}^R = Z_{j_R}^{mn}$$

→ anomaly 3-cocycle

$$\omega(m,n,k) = \Omega_{m,n}^R \Omega_{m+n,k}^R \left(U_m^\dagger \Omega_{n,k}^R (U_m^\dagger)^\dagger \Omega_{m,n+k}^R \right)^\dagger$$

$$= (-1)^{mnK}$$

$\Rightarrow$ The cohomology class of $[W]$ is nontrivial.
 Since every normalized 3-coboundary satisfies $\delta_{\mu}(1,1,1) = 1$.

$\Rightarrow U$ generates an anomalous $\mathbb{Z}_2$ symmetry.

(becomes the diagonal $\mathbb{Z}_2^M \times \mathbb{Z}_2^N$ sym in the continuum)

Remarks:

1) The anomaly of a symmetry is not a feature of a specific Hamiltonian

2) Onsite symmetries are anomaly-free

$\rightarrow$ If every $U_g = \prod_i U_i^{g_i} \Rightarrow$ every $\Omega_{g,h,k} = 1$.

3) For every GCA W , the anomaly is invariant under

$$U_g \mapsto W U_g W^\dagger \equiv \tilde{U}_g$$

$\rightarrow$ Can use $\tilde{U}_g^I = W U_g^I W^\dagger \Rightarrow \tilde{\Omega}_{g,h}^R = W \Omega_{g,h}^R W^\dagger$ which has same 3-cycle ω as $\Omega_{g,h}^R$.

4) The anomaly of $\{U_g \otimes \tilde{U}_g\}_{g \in G}$ is the product $[W][\tilde{W}]$

5) $\exists$ a 2+1d generalization (Kapustin, Xu '25 ; Kawagoe, Shirley '25)

$\rightarrow$ classified by $\mathcal{H}^*(G, U(1))$

This 1+1d anomaly on the lattice implies:

- 1) Obstruction to gauging (Seifnashri 2023)
- 2) Obstruction to trivially gapped phase (Kapustin & Sopenko 2024)
- 3) Anomaly inflow with a 2+1d SPT
(Elie & NayaK 2014, Kawagoe & Levin 2021)

General QCDs

Suppose discrete G -symmetry $\{U_g\}_{g \in G}$ with each U_g a QCD but not necessarily an FDQC

→ ie, U_g could be lattice translations.

The anomaly can be calculated using the following trick.
(Kapustin & Sopenko 2024)

- 1) Double the system: $\mathcal{H}_j \mapsto \mathcal{H}_j \otimes \mathcal{H}_j$
- 2) Consider symmetry operators

$$U_g \otimes \mathcal{U}_g^{-1}$$

For prime $\dim \mathcal{H}_j = p$, each $U_g = T^{n(g)} C_g$. Let

$$\mathcal{U}_g = T^{n(g)}$$

(Generalizer based on prime factors of $\dim \mathcal{H}_j$)

⇒ From GNVW index:

$$\mathcal{U}_g \mathcal{U}_h = \mathcal{U}_{gh}, \quad U_g \otimes \mathcal{U}_g^{-1} \text{ is an FDQC}$$

3) The $\{\gamma_g\}_{g \in G}$ symmetry is manifestly anomaly free

$\Rightarrow$ Anomaly of $\{U_g \otimes \gamma_g^{-1}\}_{g \in G}$ is the same as that of $\{U_g\}_{g \in G}$.

Example: LSM anomaly

Infinite lattice with $\mathbb{Z} \times G$ Symmetry operators:

1) Lattice translation T (not an FDQC)

2) Onsite G -symmetry operators $U_g = \prod_i U_g^{(i)}$

$\rightarrow U_g^{(i)} U_h^{(j)} = v(g, h) U_{gh}^{(j)}$ with 2-cocycle $v: G^2 \rightarrow U(1)$

$\rightarrow$ Each unit-cell associated with projective representation $[v]$.

Stacking trick:

$$T \mapsto T \otimes T^{-1}$$

$$U_g \mapsto U_g \otimes \mathbb{1}.$$

Can show that the anomaly cocycle is

$$\omega((n_1, g_1), (n_2, g_2), (n_3, g_3)) = v(g_1, g_2)^{n_3}$$

$$\Rightarrow [\omega] \neq 1 \quad \text{iff} \quad [v] \neq 1$$

$\Rightarrow$ Related to $\mathcal{H}^1(\mathbb{Z}, \mathcal{H}^2(G, U(1))) \cong \mathcal{H}^2(G, U(1))$ in the Künneth formula of $\mathcal{H}^3(\mathbb{Z} \times G, U(1))$

$\Rightarrow$ Finite-system method: Projective representation of $\{u_g\}_{g \in G}$ at L sites differs from that at $L-1$ sites by $[U]$.

Called a Lieb-Schultz-Mattis (LSM) anomaly

$\rightarrow$ Generalization of the LSM theorem.

Lecture 3

Lost two lectures: Unitary Symmetry operators, their defects, and anomalies

This lecture: generalized Symmetry on the lattice

→ Non-invertible and higher-form Symmetry

→ Currently lacks a general understanding

⇒ focus on examples.

Non-invertible Symmetry in 1+1d

Consider lattice on S^1 with L sites and for a finite group G (called G -gubits)

$$\mathcal{H}_j = \text{Span}_{\mathbb{C}} \{ |g\rangle \mid g \in G \} \quad \text{with} \quad \langle g|h \rangle = \delta_{g,h}.$$

→ $\text{Hom}(G, \text{U}(1))$ onsite Symmetry operators

$$W_{\chi} = \sum_{\vec{g}} \prod_{j=1}^L \chi(g_j) \, |\vec{g}\rangle \langle \vec{g}|$$

with $\vec{g} = (g_1, g_2, \dots, g_L)$ and $\chi \in \text{Hom}(G, \text{U}(1))$

e.g., $G = \mathbb{Z}_2$: Each $\mathcal{H}_j$ is a qubit and

$$W_{\text{triv}} = \mathbb{1} \quad W_{\text{sign}} = \prod_{j=1}^L Z_j.$$

Each χ is a 1d rep of G . What about general Unitary irrep

$$\Gamma: G \longrightarrow \text{GL}(d_{\Gamma}, \mathbb{C})$$

Let

$$W_{\Gamma} = \sum_{\vec{g}} \text{Tr}_{\Gamma} \left(\prod_{j=1}^L g_j \right) |\vec{g}\rangle \langle \vec{g}|$$

→ generalizes the W_X operators.

$$\text{Suppose } \Gamma_a \otimes \Gamma_b = \bigoplus_c N_{ab}^c \Gamma_c \quad (N_{ab}^c \in \mathbb{Z}_{\geq 0})$$

→ Follows from representation theory

$$W_{\Gamma_a} W_{\Gamma_b} = W_{\Gamma_a \otimes \Gamma_b} = \sum_c N_{ab}^c W_{\Gamma_c}.$$

$\Rightarrow \{W_{\Gamma}\}_{\Gamma \in \text{irrep}(G)}$ are $\text{Rep}(G)$ -Symmetry operators.

Non-invertible Symmetry if G is non-abelian:

$\Rightarrow$ For non-abelian G , $\exists$ an irrep Γ with $d_{\Gamma} > 1$

$\Rightarrow$ Math fact: Every Γ with $d_{\Gamma} > 1$ has at least one $g \in G$ with $\text{Tr}_{\Gamma}(g) = 0$

$\Rightarrow W_{\Gamma}$ with Γ and $d_{\Gamma} > 1$ is non-invertible.

↳ Simple statement about a Matrix!

Example: $G = S_3 = \langle r, s \mid r^3 = s^2 = 1, srs^{-1} = r^{-1} \rangle$.

→ three irreps:

1) trivial:

$$\mathbb{1}(r) = 1$$

$$\mathbb{1}(s) = 1$$

2) sign:

$$P(r) = 1$$

$$P(s) = -1$$

3) 2d:

$$E(r) = \begin{pmatrix} \cos(2\pi/3) & -\sin(2\pi/3) \\ \sin(2\pi/3) & \cos(2\pi/3) \end{pmatrix} \quad E(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

→ Symmetry operators $W_L = \mathbb{1}$, W_P , W_E satisfy

$$W_P^2 = \mathbb{1} \quad W_P W_E = W_E W_P = W_E \quad W_E^2 = \mathbb{1} + W_P + W_E$$

Non-invertibility: $\text{Tr}_E(S) = 0 \Rightarrow W_E$ is non-invertible

There are infinitely many $\text{Rep}(G)$ -symmetric local Hamiltonians.

Example:

let $\vec{X}^{(h)} |g\rangle = |hg\rangle$, $\overleftarrow{X}^{(h)} |g\rangle = |gh^{-1}\rangle$, $F|g\rangle = \delta_{g,1} |g\rangle$.

$\text{Rep}(G)$ Potts model

$$H = - \sum_{j=1}^L \sum_{h \in G} \overleftarrow{X}_j^{(h)} \vec{X}_{j+1}^{(h)} - h \sum_{j=1}^L F_j$$

→ For $G = \mathbb{Z}_q$, this is the q -state Potts model
(For $q=2$, it's the Ising model)

→ $[H, W_P] = 0$ for all P .

→ $h \ll 1$: $\text{Rep}(G)$ SSB phase

$h \gg 1$: $\text{Rep}(G)$ symmetric phase.

$\text{Rep}(G)$ symmetry defects

→ dual inrop $\bar{\Gamma}(g) = \Gamma(g)^*$

We insert a pair of $\bar{\Gamma}$ Γ defects using the truncation

$$W_P^I = \sum_{\alpha, \beta, \vec{g}} \Gamma \left(\prod_{j=\partial_L}^{\partial_R} g_j \right)_{\alpha\beta} |\alpha\rangle_L \otimes |\beta\rangle_R \otimes |\vec{g}\rangle \langle \vec{g}|$$

with $I = [\partial_L, \partial_R]$ (Natural when W_P written as an MPO)

→ Not an operator of $\mathcal{H}$ for $d_r > 1$. Instead a linear map

$$\mathcal{H} \mapsto \mathcal{H}_{\text{defect}} \cong \mathcal{H} \otimes \mathbb{C}^{d_r} \otimes \mathbb{C}^{d_r}$$

Let $(W_r^{\pm})^{\dagger} = \sum_{\alpha, \beta, \vec{g}} \Gamma^* \left(\prod_{j=j_L}^{j_R} g_j \right)_{\alpha\beta} |\vec{g}\rangle \langle \alpha|_L \otimes \langle \beta|_R \otimes \langle \vec{g}|$. The

defect insertion map satisfies

$$(W_r^{\pm})^{\dagger} W_r^{\pm} = d_r \mathbb{1}_{\mathcal{H}}.$$

→ Note: $d_r^{-1/2} W_r^{\pm}$ is an isometry

→ Spacetime picture

$$(W_r^{\pm})^{\dagger} W_r^{\pm} = \begin{array}{c} \begin{array}{c} \leftarrow \\ \boxed{\quad} \\ \rightarrow \end{array} \\ \begin{array}{c} \downarrow \\ \boxed{\quad} \\ \uparrow \end{array} \end{array} \begin{array}{c} (W_r^{\pm})^{\dagger} \\ \bullet \\ W_r^{\pm} \end{array} = d_r \mathbb{1}_{\mathcal{H}}.$$

→ Lattice version of how topological defect lines in 1+1d Euclidean QFT satisfy

$$\langle \bigcirc_a \rangle = \text{quantum dim. of } a.$$

Movement operator $M_r^{\vec{j}}$ satisfies

$$(\mathbb{1}_L \otimes M_r^{\vec{j}_R}) W_r^{\pm} = W_r^{[\vec{j}_L, \vec{j}_R+1]}$$

$$\Rightarrow M_r^{\vec{j}} = \sum_{\beta, \gamma, \vec{g}} \Gamma(g_{\vec{j}_R+1})_{\gamma\beta} |\beta\rangle_R \otimes |\vec{g}\rangle \langle \gamma|_R \otimes \langle \vec{g}|$$

→ Is a unitary operator and satisfies

$$T_{\Gamma, \text{defect}} (M_n^L M_n^{L-1} \dots M_n^1) = W_{\Gamma} \quad (\text{defect} \rightarrow \text{operator})$$

Example in $\text{Rep}(G)$ Potts model

For non-invertible symmetry defects, defect Hamiltonian found by

$$W_{\Gamma}^{\pm} H = H_n^{(\pm)} W_{\Gamma}^{\pm}$$

$$\rightarrow H_n^{(\pm)} = \mathbb{1} \otimes \mathbb{1} \otimes H + \delta H_L + \delta H_R \quad \text{with, e.g.,}$$

$$\delta H_R = - \sum_{h \in G} \mathbb{1} \otimes (\bar{\Gamma}(h) - \mathbb{1}) \otimes \overleftarrow{X}_{j_R}^{(h)} \overrightarrow{X}_{j_R+1}^{(h)}$$

$\Rightarrow$ acts nontrivially on the Γ defects Hilbert space

$\text{Rep}(G)$ -Symmetry action

Let O be a local operator with support $R \subset \Lambda$.

For $R \subset I$, The action of W_{Γ} on O is

$$O \mapsto (W_{\bar{\Gamma}}^{\pm})^{\dagger} O W_{\bar{\Gamma}}^{\pm} \equiv \alpha_{\Gamma}(O)$$

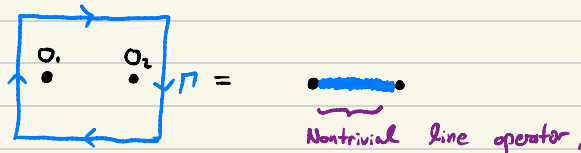
→ spacetime picture

$$(W_{\bar{\Gamma}}^{\pm})^{\dagger} O W_{\bar{\Gamma}}^{\pm} = \begin{array}{c} \begin{array}{c} \xrightarrow{\hspace{1cm}} \\ \square \\ \xleftarrow{\hspace{1cm}} \end{array} \begin{array}{c} (W_{\bar{\Gamma}}^{\pm})^{\dagger} \\ \\ W_{\bar{\Gamma}}^{\pm} \end{array} \\ \begin{array}{c} \uparrow \hspace{0.5cm} \bullet \hspace{0.5cm} \downarrow \end{array} \end{array}$$

→ e.g., $\alpha_r(\vec{X}_j^{(g)}) = \text{Tr}_r(g) \vec{X}_j^{(g)}$ } can be mapped to zero!

→ Generally, $\alpha_r(O_1 O_2) \neq \alpha_r(O_1) \alpha_r(O_2)$

⇒ spacetime picture



Remarks

1) This $\text{Rep}(G)$ example generalizes to $\text{Rep}(H)$ Symmetry with H a finite-dim semisimple Hopf algebra
(Jia '24 ; Lu, Chatterjee, Tantivasadekarn '26)

→ Anomaly-free fusion category Symmetry.

2) Defining non-invertible symmetry operators is an active topic.

3) For systems with finite $\dim \mathcal{H}$;

→ tensor product $\mathcal{H}$: only integral fusion category symmetries can be realized (Evans, Jones '25)

quantum dims $\in \sqrt{\mathbb{Z}}$ ← Weakly integral fusion cats can be realized by mixing with G&A sym

(Inamura '26 ; Wen, Inamura, Schäfer-Nameki '26)

→ general $\mathcal{H}$: Any Unitary Fusion cat sym realizable.

(Anyon chains)

4) Every unitary Fusion cat sym realizable for tensor product $\mathcal{H}$ if $\dim \mathcal{H}$ infinite (Brunner, Jones '26)

Higher-Form Symmetry

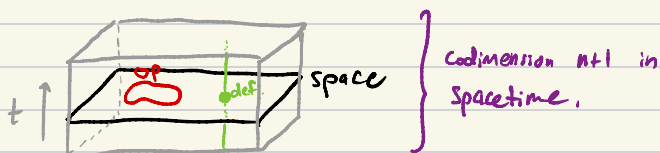
An n -form Symmetry in a $(d+1)D$ Hamiltonian lattice System has

1) topological operators with support on a $d-n$ cycle:

$$\rightarrow U(C) = U(\tilde{C}) \text{ if } C \text{ homologous to } \tilde{C}$$

2) topological defects with support on a $d-n-1$ cycle.

e.g., $d=2$ and $n=1$:



Higher-form Symmetry operators do not exist on tensor product Hilbert Spaces

Suppose $\mathcal{H} = \bigotimes_{j \in I} \mathcal{H}_j$ and C and $\tilde{C}$ have disjoint support.

$$\rightarrow U(C) = V_C \otimes \mathbb{1}_{\tilde{C}} \text{ and } U(\tilde{C}) = \mathbb{1}_C \otimes V_{\tilde{C}}$$

$$\rightarrow U(C) = U(\tilde{C}) \Rightarrow V_C \otimes \mathbb{1}_{\tilde{C}} = \mathbb{1}_C \otimes V_{\tilde{C}}$$

$$\Rightarrow V_C \propto \mathbb{1}_C \text{ and } V_{\tilde{C}} \propto \mathbb{1}_{\tilde{C}}$$

What is possible for $\bigotimes_j \mathcal{H}_j$:

$$U(C) = B_{C, \tilde{C}} U(\tilde{C})$$

with $B_{C, \tilde{C}}$ supported in region between C and $\tilde{C}$,

→ Suppose $\exists$ Mutually commuting local projectors $\{P_j\}_{j \in \Lambda}$
 s.t. $P = \prod_{j \in \Lambda} P_j$ satisfies

$$P B_{c, \tilde{c}} = B_{c, \tilde{c}} P = P \quad [P, U(c)] = 0,$$

$$\dim(\text{Im}(P)) > 0$$

$\Rightarrow$ On Subspace $\text{Im}(P) \subset \mathcal{H}$: $U(c) = U(\tilde{c})$

→ $\{U(c)\}$ are higher-form symmetry operators on $\mathcal{H}_{\text{top}} = \text{Im}(P)$

$\Rightarrow \{U(c)\}$ are "lattice higher-form" symmetry operators with respect to $\{P_j\}_{j \in \Lambda}$.

(Also called non-topological, faithful, or non-relativistic higher-form symmetry operators)

→ Lattice higher-form symmetry operator can still give rise to topological defects on a tensor-product Hilbert space.

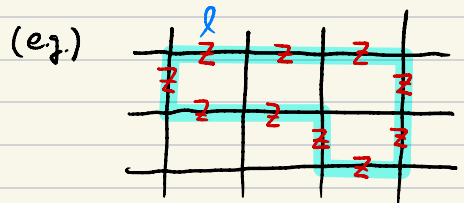
Example

Let Λ be the square lattice with a qubit on each edge

$$\mathcal{H} = \bigotimes_{e \in \Lambda} \mathbb{C}^2$$

For each lattice loop ℓ , let

$$W(\ell) = \prod_{e \in \ell} Z_e$$



→ Satisfies $\mathbb{Z}_2$ group law $W(\ell)^2 = 1$.

Denote by

Picture means take product!

$$B_p = \begin{array}{|c|} \hline \mathbb{Z} \\ \hline p \\ \hline \mathbb{Z} \\ \hline \end{array}$$

Projector to $B_p = 1$ states.

$$P_p = \frac{1}{2} (1 + B_p)$$

→ Every $W(l)$ satisfies

$$W(l + \partial r) = \left(\prod_{p \in r} B_p \right) W(l)$$

⇒ $\{W(l)\}$ are $\mathbb{Z}_2$ 1-form symmetry operators on

$$\mathcal{H}_{\text{top}} = \text{Im } \prod_p P_p = \{ |\psi\rangle \in \mathcal{H} \mid B_p |\psi\rangle = |\psi\rangle \}$$

⇒ $\{W(l)\}$ are $\mathbb{Z}_2$ lattice 1-form symmetry operators with respect to $\{P_p\}$.

The most general $\mathbb{Z}_2$ lattice 1-form symmetry operator here takes the form

$$W(\gamma) = \prod_{e \in \gamma} \mathbb{Z}_e^{\gamma_e}$$

→ $\mathbb{Z}_2$ -valued 1-cycle.

$$\gamma = \sum_e \gamma_e [e] \in \mathbb{Z}_1(\gamma; \mathbb{Z}_2)$$

→ The projector $\prod_p P_p$ implements quotient

$$\mathbb{Z}_1(\gamma; \mathbb{Z}_2) \longrightarrow H_1(\gamma; \mathbb{Z}_2)$$

Symmetric Hamiltonian: Toric code

$$H = - \sum_s A_s - \sum_p B_p \quad \text{with} \quad A_s = \begin{array}{c} \times \\ | \\ \times - s - \times \\ | \\ \times \end{array}$$

$$\longrightarrow [H, W(\gamma)] = 0 \quad \forall \quad \gamma \in \mathbb{Z}_1(\gamma; \mathbb{Z}_2)$$

→ Exactly-solvable: $\{A_s, B_p\}$ are mutually commuting.

⇒ Ground state space: $A_s = B_p = \mathbb{1}$.

⇒ $\mathcal{H}_{gs} \subset \mathcal{H}_{top}$: $\mathbb{Z}_2$ 1-form symmetry arising from $\{W(\gamma)\}_{\gamma \in \mathbb{Z}_2, (1; \mathbb{Z}_2)}$

→ can insert symmetry defect by truncating $W(\gamma)$:

$$H \xrightarrow[\text{@ site } r]{W \text{ defect}} H_r = H + 2A_r.$$

→ On full Hilbert space,

⇒ topological defect:

$$H_{r_1} = U_{12} H_{r_2} U_{12}^\dagger \quad \text{with unitary } U_{12} = \prod_{i \rightarrow j} Z.$$

Outlook of these lectures

We talked about

- 1) Unitary symmetry operators, their defects, and anomalies on the lattice
- 2) Non-invertible and higher-form symmetry on the lattice

Didn't have time for a lot of cool topics!

operator algebra framework, onsiteability, blendability, Families of lattice systems, phases of matter, gauging, fermionic lattice systems, more general crystalline symmetries and their LSM anomalies, subsystem symmetry and fractons, Anomalies of continuous unitary symmetry and generalized symmetry, Lattice higher-group symmetry, Ancillae, ...

Lots of exciting open directions!

In my view, the most exciting frontier of symmetry is symmetry on the lattice

- Use symmetry to clarify relationship between quantum lattice systems and QFTs.
- Use symmetry to characterize, classify, and constrain quantum entanglement and quantum phases of matter

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