New Lattice Fermion Anomalies from CPT

Sal Pace

IAS Princeton

IQIS Workshop: New Developments in Lattice Fermions

Shu-Heng
Shao

Elijah
Lew-Smith

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Quantum Systems

Two primary frameworks for describing quantum systems:

Quantum lattice models (QLMs)*

Quantum field theories (QFTs)

- Incredibly successful: cond-mat, hep, quant-ph, & math.

*Discrete space, continuous time

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Different frameworks, but intimately related:

$$\{\text{QLMs}\} \xrightarrow{\text{infrared limit}} \{\text{QFTs}\}$$

- Currently, more like a guiding philosophy.
- Important to make this relation precise.

*Discrete space, continuous time

Kinematics

A first step is to work at the level of kinematics:

$$\{\text{QLM symmetries}\} \rightrightarrows \{\text{QFT symmetries}\}$$

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- Some interesting questions in this arena are:
    1. What is $\{$symmetries$\}$ for QLMs and QFTs?
    2. Given a QLM symmetry, which QFT symmetries can it become?
    3. Given a QFT symmetry, which QLM symmetries can it arise from?

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  3. Given a QFT symmetry, which QLM symmetries can it arise from?

Here, we focus on invertible 0-form symmetries.

- Which QFT anomalies can arise from QLM anomalies in QLMs with Hilbert space $\bigotimes_i \mathscr{H}_i$ and finite $\dim \mathscr{H}_i$?

Anomalies

't Hooft anomalies = obstruction to gauging a global symmetry

$$Z[A + \mathrm{d}f] = Z[A]\,\mathrm{e}^{\mathrm{i}\theta(A,f)}$$

- Constrains IR dynamics

$$\text{IR theory} \neq \text{trivial theory}$$

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Anomalies are not merely a binary "yes or no" thing

- The same symmetry group can have different anomalies
- The theory of anomalies of ordinary symmetries in relativistic QFT is by now a mature subject.

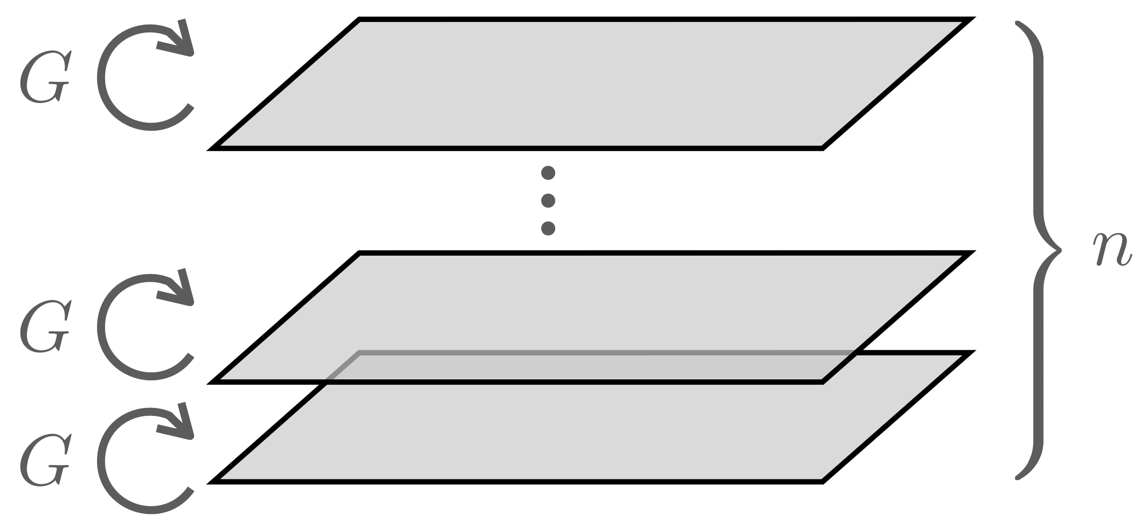
Understanding anomalies in QLMs is an increasingly active direction!

[*Bols, Carvalho, Chatterjee, Chen, Cheng, Czajka, De Roeck, De Wilde, Else, Feng, Geiko, Hsin, Ji, Kapustin, Kawagoe, Kim, Kobayashi, Levin, Liu, Long,* ***SP****, Ryu, Seiberg, Seifnashri, Shao, Shirley, Sopenko, Spodyneiko, Thorngren, Tu, Xu, Yi, Zhang, Zou,* $\cdots$]

The Order of an Anomaly

A key characterization of an anomaly ω is its order $\mathrm{ord}(\omega)$

- Smallest positive integer n such that the diagonal G symmetry of the n-copy system is anomaly-free.



- $\mathrm{ord}(\omega) = 1$ if and only if the G symmetry is anomaly free.

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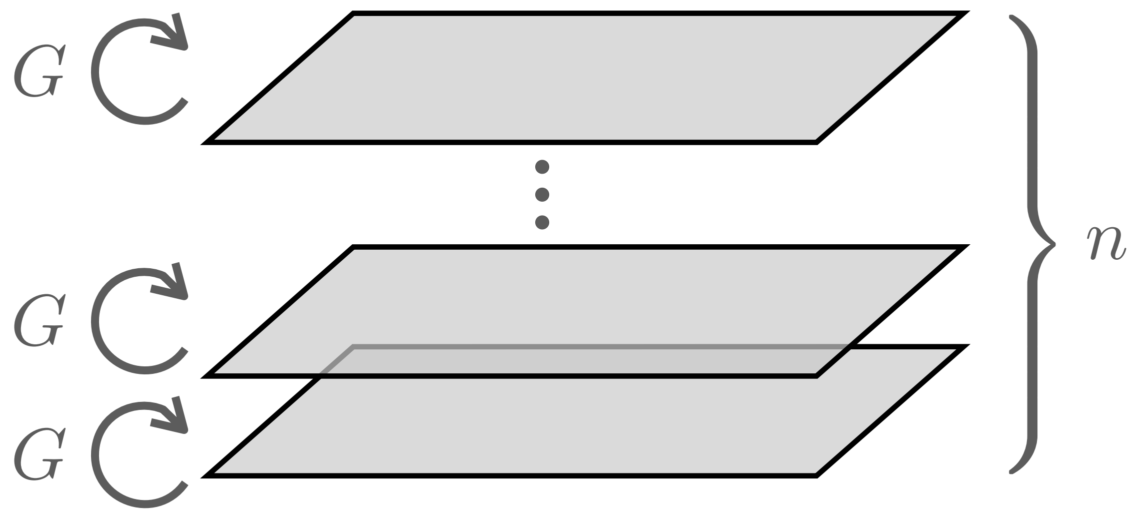

- $\mathrm{ord}(\omega) = 1$ if and only if the G symmetry is anomaly free.

If $\mathrm{ord}(\omega)$ exists $\implies$ ω is a finite-order anomaly.

- Otherwise, ω is an infinite-order anomaly: $\mathrm{ord}(\omega) = \infty$

From the UV to the IR and Back

Given a UV G_{UV} symmetry group with anomaly ω_{UV} :

Taking the IR limit induces a homomorphism

$$\rho\colon G_{\mathrm{UV}} \to G_{\mathrm{IR}}$$

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- In QFT $\Longrightarrow$ a homomorphism $\rho^*\colon \mathrm{Anom}(G_{\mathrm{IR}}) \to \mathrm{Anom}(G_{\mathrm{UV}})$ with $\mathrm{Anom}(G)$ the group of inequivalent G anomalies.

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Anomaly matching: $\rho^*(\omega_{\mathrm{IR}}) = \omega_{\mathrm{UV}}$

- A nontrivial UV anomaly implies a nontrivial IR anomaly:

$$\omega_{\mathrm{UV}} = \rho^*(\omega_{\mathrm{IR}}) \neq 0 \implies \omega_{\mathrm{IR}} \neq 0$$

- When the UV and IR symmetries are the same:

$$G_{\mathrm{UV}} \cong G_{\mathrm{IR}} \ \&\ \rho = \mathrm{id} \implies \omega_{\mathrm{IR}} = \omega_{\mathrm{UV}}$$

From the UV to the IR and Back

An anomaly's order can change under the IR limit

- The anomaly matching condition implies that

$$\mathrm{ord}(\omega_{\mathrm{UV}}) \leq \mathrm{ord}(\omega_{\mathrm{IR}})$$

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Proof for finite-order IR anomalies:

$$\begin{aligned}
&\rho^*(\omega_{\mathrm{IR}}) = \omega_{\mathrm{UV}} \\
\Longrightarrow\ & \rho^*(\omega_{\mathrm{IR}})^{\mathrm{ord}(\omega_{\mathrm{IR}})} = \omega_{\mathrm{UV}}^{\mathrm{ord}(\omega_{\mathrm{IR}})} \\
\Longrightarrow\ & \omega_{\mathrm{UV}}^{\mathrm{ord}(\omega_{\mathrm{IR}})} = 1 \\
\Longrightarrow\ & \mathrm{ord}(\omega_{\mathrm{UV}}) \mid \mathrm{ord}(\omega_{\mathrm{IR}}) \\
\Longrightarrow\ & \mathrm{ord}(\omega_{\mathrm{UV}}) \leq \mathrm{ord}(\omega_{\mathrm{IR}})
\end{aligned}$$

$$G_{\mathrm{UV}} \cong G_{\mathrm{IR}} \ \&\ \rho = \mathrm{id} \Longrightarrow \omega_{\mathrm{IR}} = \omega_{\mathrm{UV}}$$

Quantum Mechanics Example

$0+1$d example: two qubits with Hamiltonian

$$H = (3 - X_1)(1 + Z_2)$$

- $G_{\mathrm{UV}} = D_8$ generated by $A = \mathrm{CZ}_{1,2}$, $B = X_1$, and $C = Z_2$:

$$A^2 = B^2 = C^2 = 1, \qquad AB = CBA.$$

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The IR (ground state space) is the $Z_2 = -1$ eigenspace

$$A_{\mathrm{IR}}^2 = B_{\mathrm{IR}}^2 = 1, \qquad A_{\mathrm{IR}} B_{\mathrm{IR}} = -B_{\mathrm{IR}} A_{\mathrm{IR}}$$

$\implies$ $G_{\mathrm{IR}} \supset \mathbb{Z}_2 \times \mathbb{Z}_2$ with a nontrivial anomaly

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- $\rho \colon D_8 \to G_{\mathrm{IR}}$ with $\rho(D_8) = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\ker(\rho) = \langle C \rangle \cong \mathbb{Z}_2$
- Emergent anomaly: $\omega_{\mathrm{IR}} \neq 0$ but $\rho^*(\omega_{\mathrm{IR}}) = \omega_{\mathrm{UV}} = 0$

$$\mathrm{ord}(\omega_{\mathrm{UV}}) = 1 \leq \mathrm{ord}(\omega_{\mathrm{IR}}) = 2$$

tl;dr

Infinite-order anomalies are elusive in QLMs.

- No-go theorems for compact locality-preserving symmetries

  [*Kapustin, Sopenko '24; Liu '26*]

- In contrast, infinite-order anomalies are ubiquitous in QFT (e.g., perturbative anomalies) and imply gaplessness

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Here, we consider $1+1$d fermionic QLMs and show [*Lew-Smith, **SP**, Shao '26*]

QLM	UV Symmetry	IR Symmetry	Anomaly Order UV $\to$ IR
Staggered Fermion	Onsager [*Chatterjee, **SP**, Shao '24*]	$\dfrac{\mathrm{U}(1)_{\mathrm{V}} \times \mathrm{U}(1)_{\mathrm{A}}}{\mathbb{Z}_2}$	$2 \to \infty$
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Lattice fermions as a playground for conceptual questions about anomalies in QLMs and QFTs

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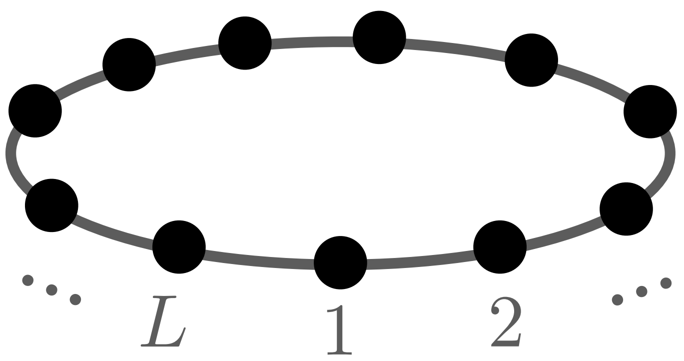
A Simple Lattice Model

Consider a finite lattice on a ring with even L sites



Fermion operator c_j at each site j

$$\{c_j, c_{j'}^\dagger\} = \delta_{j,j'} \qquad \{c_j, c_{j'}\} = 0$$

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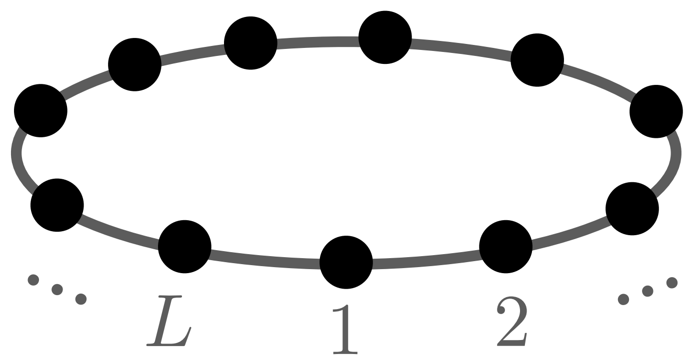

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Staggered fermion Hamiltonian $H = \mathrm{i}\sum_{j=1}^{L}\left(c_j^\dagger c_{j+1} - c_{j+1}^\dagger c_j\right)$

[*Banks, Susskind, Kogut '76*]

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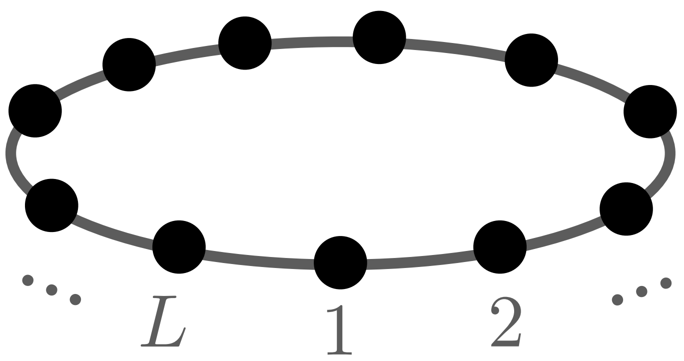

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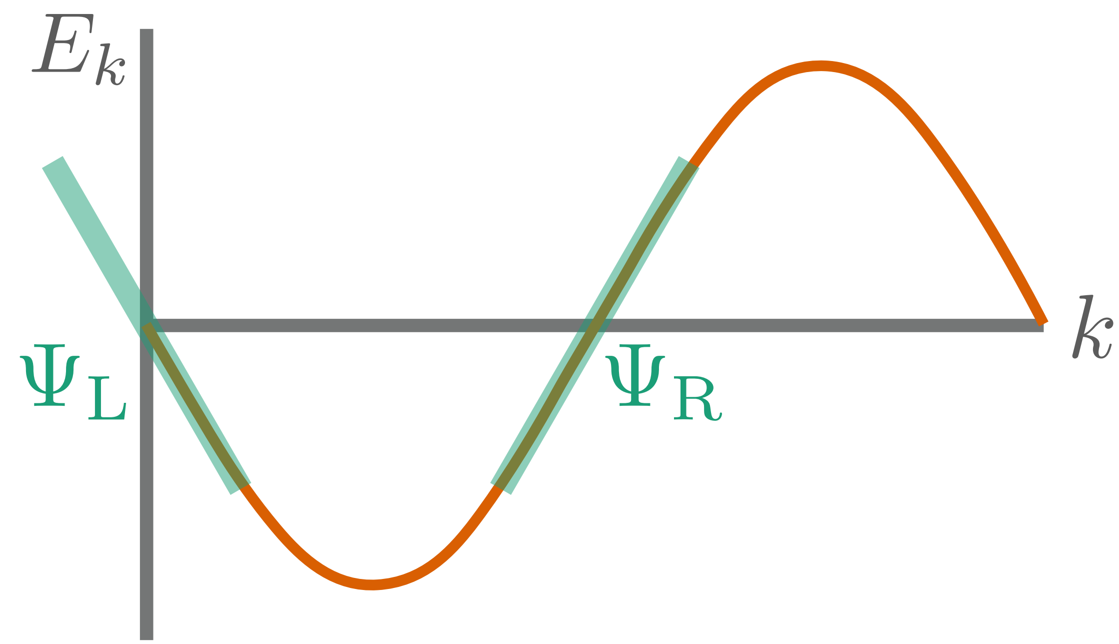
- H becomes a free, massless Dirac fermion theory in IR
- In momentum space

$$H = \sum_{k \in \mathrm{BZ}} E_k \, c^\dagger_k c_k$$

$$E_k = -2\sin(k)$$



Onsager Symmetry I: What is it

The symmetry of 1+1d massless Dirac fermion includes

$$\mathrm{U}(1)_{\mathrm{V}} : \Psi \mapsto \mathrm{e}^{-\mathrm{i}\theta_{\mathrm{V}}} \Psi \qquad \mathrm{U}(1)_{\mathrm{A}} : \Psi \mapsto \mathrm{e}^{-\mathrm{i}\theta_{\mathrm{A}} \sigma^z} \Psi$$

Does this IR symmetry arise from a UV symmetry of H?

- i.e., G_{UV} such that $\rho(G_{\mathrm{UV}}) \cong [\mathrm{U}(1)_{\mathrm{V}} \times \mathrm{U}(1)_{\mathrm{A}}]/\mathbb{Z}_2$

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Yes! Let $a_j = c_j^\dagger + c_j$ and $b_j = \mathrm{i}(c_j^\dagger - c_j)$

➤ Conserved, quantized charges

$$Q_0 = \frac{\mathrm{i}}{2}\sum_{j=1}^{L} a_j b_j \qquad Q_1 = \frac{\mathrm{i}}{2}\sum_{j=1}^{L} a_j b_{j+1}$$

have the IR limit [*Chatterjee, **SP**, Shao '24*]

$$Q_0 \xrightarrow{\mathrm{IR}} \mathcal{Q}_{\mathrm{V}} \qquad Q_1 \xrightarrow{\mathrm{IR}} \mathcal{Q}_{\mathrm{A}}$$

Onsager Symmetry I: What is it

Q_0 and Q_1 generate the Onsager algebra* [*Onsager '44*]

$$Q_n = \frac{\mathrm{i}}{2}\sum_{j=1}^{L} a_j b_{j+n} \qquad G_n = \frac{\mathrm{i}}{2}\sum_{j=1}^{L}(a_j a_{j+n} - b_j b_{j+n})$$

$$[Q_n, Q_m] = \mathrm{i}G_{m-n} \qquad [G_n, G_m] = 0$$

$$[Q_n, G_m] = 2\mathrm{i}(Q_{n-m} - Q_{n+m})$$

UV to IR symmetry map

$$Q_n \xrightarrow{\text{IR}} \begin{cases} \mathcal{Q}_{\mathrm{V}} & n \text{ even} \\ \mathcal{Q}_{\mathrm{A}} & n \text{ odd} \end{cases} \qquad G_n \xrightarrow{\text{IR}} 0$$

*Strictly speaking, this is the Onsager algebra only when $L \to \infty$

$$Q_0 \xrightarrow{\text{IR}} \mathcal{Q}_{\mathrm{V}} \qquad Q_1 \xrightarrow{\text{IR}} \mathcal{Q}_{\mathrm{A}}$$

Onsager Symmetry II: Its Anomaly

The Dirac fermion's $\left[\mathrm{U}(1)_{\mathrm{V}} \times \mathrm{U}(1)_{\mathrm{A}}\right]/\mathbb{Z}_2$ symmetry is anomalous

- $S_{\text{inflow}} = \dfrac{\mathrm{i}}{\pi} \displaystyle\int A_{\mathrm{V}} \mathrm{d} A_{\mathrm{A}} \implies$ infinite-order anomaly

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What even is an anomaly in a QLM?

Here, we view an anomaly as a robust obstruction to a trivial symmetric gapped phase

- Avoids having to define gauging
- Sometimes called an LSM-type anomaly
- Commonly used for non-invertible symmetry

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Is the Onsager symmetry anomalous? Yes! [*Chatterjee, **SP**, Shao '24*]

- Can prove each symmetric local Hamiltonian takes the form

$$H_g = \mathrm{i} \sum_n \sum_{j=1}^{L} g_n \left(a_j a_{j+n} + b_j b_{j+n}\right),$$

  which never has a unique gapped ground state

- Each $H_g \neq 0$ is gapless $\implies$ symmetry-enforced gaplessness

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In the staggered fermion Hamiltonian, the Dirac fermion's chiral anomaly arises from the Onsager symmetry's lattice anomaly.

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What is the order of the Onsager symmetry's anomaly?

It is order two: The diagonal Onsager symmetry of the 2-copy system is anomaly-free [*Lew-Smith, **SP**, Shao '26*]

- 2-copy system has two fermion operators c_j^{I}, c_j^{II} per site j.
- Diagonal Onsager symmetry generated by

$$Q_0^{\mathrm{diag}} = Q_0^{\mathrm{I}} + Q_0^{\mathrm{II}}, \qquad Q_1^{\mathrm{diag}} = Q_1^{\mathrm{I}} + Q_1^{\mathrm{II}}$$

- A symmetric Hamiltonian with a unique gapped ground state:

$$H_{\mathrm{gap}} = \frac{\mathrm{i}}{2} \sum_{j=1}^{L} \left(a_j^{\mathrm{I}} a_j^{\mathrm{II}} + b_j^{\mathrm{I}} b_j^{\mathrm{II}} \right)$$

$\Longrightarrow$ The $(Q_0^{\mathrm{diag}}, Q_1^{\mathrm{diag}})$ Onsager symmetry is anomaly-free

UV to IR Anomaly Order

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Deformed 2-copy staggered fermion Hamiltonian

$$\frac{\mathrm{i}}{2} \sum_{j=1}^{L} \left(a_j^{\mathrm{I}} a_{j+1}^{\mathrm{I}} + b_j^{\mathrm{I}} b_{j+1}^{\mathrm{I}} + a_j^{\mathrm{II}} a_{j+1}^{\mathrm{II}} + b_j^{\mathrm{II}} b_{j+1}^{\mathrm{II}} \right) + g H_{\mathrm{gap}}$$

- Single-particle dispersions $E_k^{\pm} = -2\sin(k) \pm g$

IR limit of H_{gap} deformation ($g \ll 1$)

$$i\mu (\Psi_{\mathrm{L}}^{\mathrm{I}})^\dagger \Psi_{\mathrm{L}}^{\mathrm{II}} + i\mu (\Psi_{\mathrm{R}}^{\mathrm{I}})^\dagger \Psi_{\mathrm{R}}^{\mathrm{II}} + \text{h.c.}$$

- Chemical potential term

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Can the order of the Onsager symmetry's anomaly be enhanced to infinite order?

- One way to increase an anomaly's order is by introducing additional symmetry

Here, we will add a lattice CPT symmetry

- Natural symmetry to add from the QFT perspective

Lattice CPT Symmetry

Consider the anti-unitary operator Π satisfying

$$\Pi^2 = 1, \qquad \Pi\,\mathrm{i}\,\Pi^\dagger = -\mathrm{i}, \qquad \Pi\, c_j\, \Pi^\dagger = c^\dagger_{-j+1}$$

- Is a $\mathbb{Z}_2$ anti-unitary reflection operator

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- Is a $\mathbb{Z}_2$ anti-unitary reflection operator

It is a symmetry of the staggered fermion Hamiltonian

$$H = \mathrm{i}\sum_{j=1}^{L}\left(c_j^\dagger c_{j+1} - c_{j+1}^\dagger c_j\right)$$

- In the IR, Π becomes the CPT symmetry operator of the Dirac fermion QFT

  $\implies \Pi$ is a lattice $\mathbb{Z}_2^{\mathrm{CPT}}$ symmetry operator

Lattice CPT Symmetry

Consider the anti-unitary operator Π satisfying

Π interplays with the Onsager charge operators:

$$\Pi Q_n \Pi^\dagger = -Q_{-n}, \qquad \Pi G_n \Pi^\dagger = G_n$$

- (Q_0, Q_1, Π) form the symmetry group

$$\text{Onsager} \rtimes \mathbb{Z}_2^{\text{CPT}}$$

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➤ (Q_0, Q_1, Π) form the symmetry group

$$\text{Onsager} \rtimes \mathbb{Z}_2^{\text{CPT}}$$

➤ In the IR of the staggered fermion Hamiltonian

$$\Pi Q_n \Pi^\dagger = -Q_{-n} \xrightarrow{\text{IR}} \begin{cases} \Pi_{\text{IR}} \mathcal{Q}_{\text{V}} \Pi_{\text{IR}}^\dagger = -\mathcal{Q}_{\text{V}} & n \text{ even} \\ \Pi_{\text{IR}} \mathcal{Q}_{\text{A}} \Pi_{\text{IR}}^\dagger = -\mathcal{Q}_{\text{A}} & n \text{ odd} \end{cases}$$

$$\text{Onsager} \rtimes \mathbb{Z}_2^{\text{CPT}} \xrightarrow{\text{IR}} [\text{U}(1)_{\text{V}} \times \text{U}(1)_{\text{A}}]/\mathbb{Z}_2 \times \mathbb{Z}_2^{\text{CPT}}$$

Onsager + CPT: Anomaly Order

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It's an infinite-order anomaly: The diagonal symmetry of the N-copy system is anomalous for all N. [*Lew-Smith, **SP**, Shao '26*]

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- N-copy system has N fermion operators $\{c_j^{(\ell)}\}_{\ell=1}^N$ per site j
- Diagonal Onsager $\rtimes \mathbb{Z}_2^{\mathrm{CPT}}$ symmetry generated by

$$Q_0^{\mathrm{diag}} = \sum_{\ell=1}^{N} Q_0^{(\ell)}, \qquad Q_1^{\mathrm{diag}} = \sum_{\ell=1}^{N} Q_1^{(\ell)}, \qquad \Pi^{\mathrm{diag}}$$

- Can prove each symmetric local Hamiltonian takes the form

$$\frac{\mathrm{i}}{2} \sum_{n} \sum_{I,J=1}^{N} \sum_{j=1}^{L} g_n^{IJ} \left(a_j^I a_{j+n}^J + b_j^I b_{j+n}^J \right) \qquad (g_{-n}^{IJ} = -g_n^{IJ})$$

which never has a unique gapped ground state

Recap

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Recap

QLM	UV Symmetry	IR Symmetry	Anomaly Order $\mathrm{UV} \to \mathrm{IR}$
Staggered Fermion	Onsager	$\dfrac{\mathrm{U}(1)_{\mathrm{V}} \times \mathrm{U}(1)_{\mathrm{A}}}{\mathbb{Z}_2}$	$2 \to \infty$
	Onsager $\rtimes \mathbb{Z}_2^{\mathrm{CPT}}$	$\dfrac{\mathrm{U}(1)_{\mathrm{V}} \times \mathrm{U}(1)_{\mathrm{A}}}{\mathbb{Z}_2} \times \mathbb{Z}_2^{\mathrm{CPT}}$	$\infty \to \infty$

Recap

QLM	UV Symmetry	IR Symmetry	Anomaly Order UV $\to$ IR
Staggered Fermion	Onsager	$\dfrac{U(1)_V \times U(1)_A}{\mathbb{Z}_2}$	$2 \to \infty$
	$\text{Onsager} \rtimes \mathbb{Z}_2^{\text{CPT}}$	$\dfrac{U(1)_V \times U(1)_A}{\mathbb{Z}_2} \times \mathbb{Z}_2^{\text{CPT}}$	$\infty \to \infty$
Heisenberg Chain	$\mathbb{Z}^{\text{tran}}$	$\mathbb{Z}_2$	$1 \to 2$ [*Metlitski, Thorngren '17*]
	$\mathbb{Z}^{\text{tran}} \rtimes \mathbb{Z}_2^{\text{CPT}}$	$\mathbb{Z}_2 \times \mathbb{Z}_2^{\text{CPT}}$	$2 \to 2$ [*Seiberg, Shao, Zhang '25*]
Kitaev Chain	$\mathbb{Z}^{\text{maj}}$	$\mathbb{Z}_2$	$2 \to 8$ [*Seiberg, Shao '23*]
	$\mathbb{Z}^{\text{maj}} \rtimes \mathbb{Z}_2^{\text{CPT}}$	$\mathbb{Z}_2 \times \mathbb{Z}_2^{\text{CPT}}$	$8 \to 8$ [*Seiberg, Shao, Zhang '25*]

Future Directions

The role of CPT, and more generally CRT, in QLMs

- Interesting results also noted for invertible phases [*Ogata '22; Sopenko '25*]

Systematic study of the anomaly's stability to ancillas

- Preliminary results: the anomaly is robust to a natural class of nononsite ancillas [*Lew-Smith, **SP**, Shao '26*]

Analogous story for Onsager symmetries in $> 1+1$d

- The anomalies of such Onsager symmetries can realize parity anomalies [***SP**, Kim, Chatterjee, Shao '25*], Witten's SU(2) anomaly [*Gioia, Thorngren '25*], and symmetry-enforced Fermi surfaces [*Kim, **SP**, Shao '25*].