Comments on Clifford QCA Symmetries

(Work in progress with Shu-Heng Shao)

Consider a 1+1d quantum lattice system:

→ QM system $\mathcal{H}_j$ @ site j with $\dim \mathcal{H}_j$ = finite.

→ Total system: (Forms spatial lattice)

$$\begin{array}{ccccc} & \mathcal{H}_{j-1} & \mathcal{H}_j & \mathcal{H}_{j+1} & \\ \cdots & \bullet & \bullet & \bullet & \cdots \end{array}$$

Hilbert space of finite region Γ is $\bigotimes_{j\in\Gamma} \mathcal{H}_j$

eg) qubit system: $\mathcal{H}_j = \mathbb{C}^2$

$\Rightarrow$ onsite operator algebra: $A_j = M_2(\mathbb{C}) = \mathrm{Span}_{\mathbb{C}}\{1, X, Y, Z\}$

What is a symmetry operator? (Nat's talk)

Strictly locality-preserving unitary symmetries $\Rightarrow$ a QCA α

→ α is a $*$-automorphism of the operator algebra st $\alpha(O_j)$ has support on $\{j-R, j-R+1, \dots j+R\}$ (R = range of α)

→ Finite lattice $\Rightarrow$ $\alpha(O_j) = U_\alpha O_j U_\alpha^\dagger$ for unitary U_α.

Examples in qubit systems (assume finite lattice)

	U_α	QCA?
onsite FDQC	$\prod_j X_j$	Yes

nononsite FDQC	$\prod_j CZ_{j,j+1}$ ($\frac{1}{2}(1+Z_j+Z_{j+1}-Z_j Z_{j+1})$)	Yes
nononsite, non-FDQC	Lattice translations	Yes
	e^{itH} (Finite time evolution)	No (locality preserving with tails)

Question: How much of the QCA landscape is compatible with our usual perspectives of symmetry
eg) gauging, anomalies, onsiteability

Consider simple scenario:

1) qubit system with Clifford QCA sym ops

$\rightsquigarrow$ a QCA that maps Pauli strings $\left(i^{n} \prod_j X_j^{a_j} Z_j^{b_j} \right)$ to Pauli strings.

2) Cyclic sym group $G = \langle \alpha \rangle = \mathbb{Z}_n$ or $\mathbb{Z}$ → Interesting case.

Example: $\alpha(X_j) = Z_j$ $\qquad$ $\alpha(Z_j) = Z_{j-1} X_j Z_{j+1}$

→ unitary $U_\alpha = \prod_j CZ_{j,j+1} \prod_j H_j$ ⇒ clearly a FDQC.

→ Hadamard gate $\frac{1}{\sqrt{2}}(X_j + Z_j)$

→ Operator spreading under α^n (ignoring overall phase)

n	
	$\vdots$
3	$Z \; X \; Z \; X \; Z$
2	$Z \; X \; Z$
1	Z
0	X

Won't show proofs. It all matches intuition.

→ Can prove the following:

1)
$$\langle \alpha \rangle = \begin{cases} \mathbb{Z} & \text{infinite lattice,} \\ \mathbb{Z}_L & \text{even } L \text{ sites,} \\ \mathbb{Z}_{4L} & \text{odd } L \text{ sites.} \end{cases}$$

2) There are no local symmetric operators. (besides $\mathbb{1}$)

→ like lattice translations, but α is a FDQC.

3) There are gliders (local operator O_j s.t. $\alpha(O_j) = O_{j+v}$)

glider	v
$R_j = X_j Z_{j+1}$	1
$L_j = Z_{j-1} X_j$	-1

→ all gliders constructed from L_j, R_j.

4) Each symmetric local Hamiltonian takes the form

$$H_L(\{L_j\}) + H_R(\{R_j\})$$

with $H_{L,R}$ translation-invariant

→ Example: $H = \sum_j L_j + \sum_j R_j = \sum_j (Z_j X_{j+1} + X_j Z_{j+1})$

⇒ Unitarily equivalent to $\sum_j (X_j X_{j+1} + Y_j Y_{j+1})$

⇒ IR is $R = \sqrt{2}$ compact boson. $\langle \alpha \rangle$ becomes $\mathbb{Z}_2^c$

↳ A reason to care!

Interesting: No anomalous 1+1d $\mathbb{Z}$ sym.

5) No symmetric gapped Hamiltonian with unique ground state

→ Every Symmetric H also commutes with $Y = \prod_j Y_j$, and $\langle Y \rangle \times \langle \alpha \rangle = \mathbb{Z}_2 \times \mathbb{Z}$ has an anomaly.

Expectation: There is an SPT Hamiltonian after tensoring in ancillas. May require ∞-dim ancillas.

6) α is not onsiteable bc gliders

7) Conjecture: $\langle \alpha \rangle$ cannot be gauged

→ Intuition: no local symmetric opperators
(More detail: obstruction to find commuting gauss operators)

Generalizations

There are 3 types of translation-invariant cQCAs
(Gütschow et al 0906.3195)

1) Periodic cQCA ⇒ $\alpha^n = 1$.

2) glider cQCA ⇒ α has gliders

3) fractal cQCA ⇒ α^n transformation is diffusive.

(generic transl-inv cQCA is a fractal cQCA)

→ Example: $\alpha(X_j) = Z_j \qquad \alpha(Z_j) = Z_{j-1} Y_j Z_{j+1}$

$$U_\alpha = \prod_j CZ_{j,j+1} \prod_j S_j H_j \qquad S_j = e^{i\frac{\pi}{4}(1-Z_j)}, \quad \begin{array}{l} X_j \longmapsto Y_j \\ Z_j \longmapsto Z_j \end{array}$$

$\langle \alpha \rangle = \mathbb{Z}$ and $H = \text{const.}$ is only sym Hamiltonian.

	periodic	glider	fractal
Sym group on inf. lat.	$\mathbb{Z}_N$	$\mathbb{Z}$	$\mathbb{Z}$
Symmetric local ops.?	Yes	Sometimes	No
Gauge-able?	Sometimes	Sometimes?	No?
Onsiteable?	Sometimes	No	No
$\exists$ nontrivial sym Ham?	Yes	Yes	No
$\exists$ SPT Hamiltonian?	Sometimes	Sometimes	No