Entangled weak SPTs from projective non-invertible symmetries

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SP, Lam, Aksoy arXiv:2409.18113
[SciPost Phys. 18, 028 (2025)]

Quantum phases and symmetry

A fundamental problem in CMT/QFT/Math-ph is to understand quantum phases

1. How do we diagnose different quantum phases?
2. What are the allowed possible quantum phases?

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Sometimes, phases are characterized by a symmetry

- Superfluids by U(1) boson number conservation
- Topological insulators by $U(1)_f$ and time-reversal

For such phases, symmetries provide answers to questions (1) and (2).

Generalized symmetries

There has been a recent flurry of interest in generalizing the notion of symmetries

- Ordinary symmetries transform local operators in an invertible manner (e.g., $c_r^\dagger \to \mathrm{e}^{\mathrm{i}\theta} c_r^\dagger$)

- So-called generalized symmetries modify this definition

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Non-invertible symmetries have non-invertible transformations

[Bhardwaj, Tachikawa '17; Chang, Lin, Shao, Wang, Yin '18; ⋯]

- Can arise at critical points from Kramers-Wannier dualities
  [Thorngren, Yang '21 ; Choi, Córdova, Hsin, Lam, Shao '21; ⋯]
- Can emerge in ordered phases (are symmetries of nonlinear sigma models) [Chen, Tanizaki '22; Hsin '22; **SP** '23; **SP**, Zhu, Beaudry, X-G Wen '23]

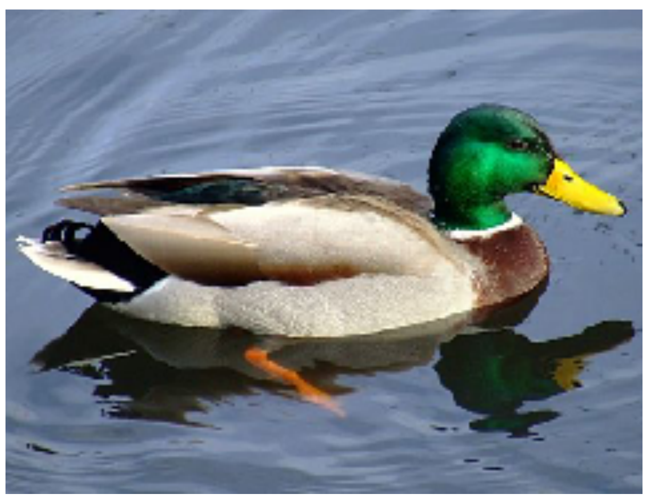
Generalized symmetries

Q: Why should we consider these as symmetries?

Generalized symmetries

Q: Why should we consider these as symmetries?

A: They pass the duck test!



If it looks like a duck, swims like a duck, and quacks like a duck, then it probably is a duck.

- Have conservation laws
- Can constrain phase diagrams (be anomalous)
- Can characterize SSB and SPT phases

Quantum phases + generalized symmetry

Which quantum phases are characterized by generalized symmetries?

Quantum phases + generalized symmetry

Which quantum phases are characterized by generalized symmetries?

Systematic build-a-phase recipe

(1) Choose your generalized symmetries adjectives

$$a_1 – a_2 – a_3 – \cdots \text{ Symmetry}$$

(2) Specify "SSB and SPT pattern"

Quantum phases + generalized symmetry

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Ordered phases *Topological insulators*

Topological order *Maxwell phases* *Higgs phases* *Fracton phases*

Phases we have yet to name!

Quantum phases + generalized symmetry

Which quantum phases are characterized by generalized symmetries?

Why care?

1. Provides a novel and unifying perspective of quantum phases
2. Guides us towards new quantum phases and models
3. Further develops a classification of quantum phases based on symmetries (a "generalized Landau paradigm")

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Quantum phases + generalized symmetry

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Wh

1.

2.

3. d

on symmetries (a "generalized Landau paradigm")

We are making incredible progress!

[Albert, Aksoy, Atinucci, Barkeshli, Bhardwaj, Bottini, Bulmash, Burnell, Cao, Chatterjee, Chen, Cheng, Choi, Copetti, Córdova, Delcamp, Delfino, Devakul, Dua, Dumitrescu, Eck, Fechisin, Fendley, Gai, Gaiotto, Garre-Rubio, Gorantla, Gu, Han, Hsin, Huang, Inamura, Ji, Jia, Jian, Kapustin, Kobayashi, Kong, Lake, Lam, Lan, Lee, Li, Litvinov, Liu, Lootens, Ma, Meng, Molnár, Myerson-Jain, Nandkishore, Oh, Ohmori, Pajer, Pichler, Prem, Rayhaun, Sanghavi, Schäfer-Nameki, Seiberg, Seifnashri, Shao, Sondhi, Stahl, Stephen, Tantivasadakarn, Thorngren, Tiwari, Tsui, Ueda, Verresen, Verstraete, Vijay, Wang, Warman, Wen, Willet, Williamson, Wu, Xu, Yamazaki, Yan, Yang, Yoshida, Zhang, Zheng, ⋯]

Here: focus on beyond-relativistic-QFT-symmetries

TL;DR for this talk

This talk: $1+1$D SPT phases characterized by lattice translation and non-invertible symmetries

- Find a new class of entangled weak SPTs characterized by projective non-invertible symmetries on the lattice

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<u>Outline</u>

1. Weak SPTs from a symmetry defect perspective
2. Simple example of an entangled weak SPT characterized by a projective non-invertible symmetry
3. General discussion on projective $Z(G)\times \mathsf{Rep}(G)$ symmetry and (SPT-)LSM theorems

Recap: SPTs and symmetry defects

Recall: An SPT phase is a gapped quantum phase protected by a symmetry with a unique ground state on all closed spatial manifolds [Chen, Gu, Liu, Wen 2011; ⋯]

- SPTs are characterized by their bulk response to static probes: Background gauge fields and symmetry defects

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Recall: Symmetry defects are localized modifications to the Hamiltonian $H^{(\Sigma)}_{\text{defect}} = H + \delta H(\Sigma)$ and other operators

- Moved using unitary operators (are topological defects)
- Twisted boundary conditions $(T_\perp)^L =$ Symmetry operator

Example: $\mathbb{Z}_2$ weak SPTs

SPTs can be protected by internal and spacetime symmetries

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SPTs can be protected by internal and spacetime symmetries

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Example: 1d periodic lattice with a qubit on each site $j \sim j + L$

$$H_+ = -\sum_j X_j \quad \text{vs.} \quad H_- = +\sum_j X_j$$

- Both have a unique gapped ground state $|\mathrm{GS}_\pm\rangle = \otimes_j |\pm\rangle$
- Symmetries: $\mathbb{Z}_2 \times \mathbb{Z}_L$ with $U = \prod_j X_j$ and $T: j \to j+1$

H_+ and H_- are both in $\mathbb{Z}_2$ weak SPT phases

Example: $\mathbb{Z}_2$ weak SPTs

Are H_+ and H_- in different $\mathbb{Z}_2$ weak SPT phases?

Let's insert a $U = \prod_j X_j$ symmetry defect at $\langle L,1 \rangle$

- Neither H_+ or H_- are modified by $Z_{j+L} = -Z_j$
- Translation operator becomes $T = X_1 T_{\text{defect-free}}$ $(T^L = U)$

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	Even L	Even L, $\mathbb{Z}_2$ symmetry defect
$U\vert\mathrm{GS}_\pm\rangle =$	$+\vert\mathrm{GS}_\pm\rangle$	$+\vert\mathrm{GS}_\pm\rangle$
$T\vert\mathrm{GS}_\pm\rangle =$	$+\vert\mathrm{GS}_\pm\rangle$	$\pm\vert\mathrm{GS}_\pm\rangle$

Different $\mathbb{Z}_2$ weak SPTs

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Inserting a translation defect is done by

$$T^L = 1 \to T^L = T \implies L \to L - 1$$

- Translation defect carries $\mathbb{Z}_2$ symmetry charge in $|\text{GS}_-\rangle$

Weak SPTs $\leftrightarrow$ Translation defects dressed by SPTs

	Even L	Even L, $\mathbb{Z}_2$ symmetry defect	Odd L
$U\|\text{GS}_\pm\rangle =$	$+\|\text{GS}_\pm\rangle$	$+\|\text{GS}_\pm\rangle$	$\pm\|\text{GS}_\pm\rangle$
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A curious Hamiltonian

1d periodic lattice with a single qubit and $\mathbb{Z}_4$ qudit on each site $j \sim j + L$ [**SP**, Lam, Aksoy '24]

- σ^x, σ^z act on qubits: $(\sigma^x)^2 = (\sigma^z)^2 = 1$ and $\sigma^z \sigma^x = -\sigma^x \sigma^z$
- X, Z act on $\mathbb{Z}_4$ qudits: $X^4 = Z^4 = 1$ and $ZX = \mathrm{i}\, XZ$

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$$H = \sum_j (Z_j - Z_j^\dagger)\, \sigma_j^z\, (Z_{j+1} - Z_{j+1}^\dagger) - \sigma_j^x\, C_{j+1}\, \sigma_{j+1}^x$$

- C acts as $X \to X^\dagger$ and $Z \to Z^\dagger$
- Is a sum of commuting terms and has a unique gapped ground state

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- C ... X ... $X^\dagger$ ... Z ... $Z^\dagger$
- Is ... gr...

$$|\mathrm{GS}\rangle = \sum_{\substack{\{\varphi_j = 0,1\}\\ \{\alpha_j = 0,2\}}} \mathrm{i}^{\sum_j \alpha_j(\varphi_j - \varphi_{j-1})} \bigotimes_j |\sigma^x_j = (-1)^{\varphi_j},\ Z_j = \mathrm{i}^{\alpha_j+1}\rangle$$

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$$H = \sum_j (Z_j - Z_j^\dagger)\, \sigma_j^z\, (Z_{j+1} - Z_{j+1}^\dagger) - \sigma_j^x\, C_{j+1}\, \sigma_{j+1}^x$$

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- $\mathbb{Z}_L$ lattice translations T: $j \to j + 1$
- Three $\mathbb{Z}_2$ symmetry operators

$$U = \prod_j X_j^2, \qquad R_1 = \prod_j \sigma_j^z, \qquad R_2 = \prod_j Z_j^2$$

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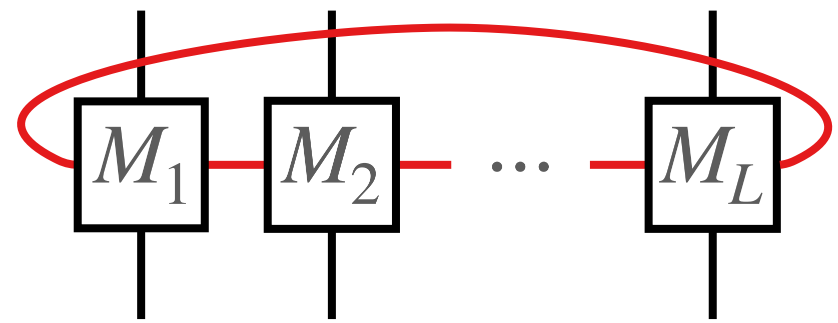
- 🤷‍♂️ symmetry operator

$$R_{\mathsf{E}} = \frac{1}{2}\left(1 + R_1\right)\left(1 + R_2\right)\prod_j Z_j^{\prod_{k=1}^{j-1}\sigma_k^z}$$

Some curious symmetries

R_{E} can be written as a $\chi = 2$ matrix product operator

$$R_{\mathsf{E}} = \mathrm{Tr}\Big(\prod_{j=1}^{L} M_j\Big) \equiv$$



- MPO tensor

$$M_j = \frac{1}{2}\begin{pmatrix} Z_j + Z_j^\dagger & \mathrm{i}\,(Z_j - Z_j^\dagger)\,\sigma_j^z \\ -\mathrm{i}\,(Z_j - Z_j^\dagger) & (Z_j + Z_j^\dagger)\,\sigma_j^z \end{pmatrix}$$

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$$H = \sum_j (Z_j - Z_j^\dagger)\, \sigma_j^z\, (Z_{j+1} - Z_{j+1}^\dagger) - \sigma_j^x\, C_{j+1}\, \sigma_{j+1}^x$$

What are the symmetries of H?

R_E is a non-invertible symmetry operator

- $R_1 \,|\psi\rangle = -\,|\psi\rangle$ or $R_2\,|\psi\rangle = -\,|\psi\rangle \implies R_\mathsf{E}\,|\psi\rangle = 0$
- R_E have zero-eigenvalues $\implies R_\mathsf{E}$ is non-invertible

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These symmetry operators obey

$$U^2 = 1, \quad R_i^2 = 1, \quad R_{\mathsf{E}}^2 = 1 + R_1 + R_2 + R_1 R_2, \quad R_{\mathsf{E}} R_i = R_i R_{\mathsf{E}} = R_{\mathsf{E}}$$

$$U\, R_{\mathsf{E}} = (-1)^L R_{\mathsf{E}}\, U$$

➤ Form a (projective) $\mathbb{Z}_2 \times \mathsf{Rep}(D_8)$ symmetry*

Dihedral group of order 8 $D_8 \simeq \langle r, s \mid r^2 = s^4 = 1, rsr = s^3 \rangle$

➤ Four 1d reps $\mathbf{1}$, $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3 = \mathbf{P}_1 \otimes \mathbf{P}_2$ and one 2d irrep E

$$\mathbf{P}_i \otimes \mathbf{P}_i = \mathbf{1} \qquad \mathsf{E} \otimes \mathsf{E} = \mathbf{1} \oplus \mathbf{P}_1 \oplus \mathbf{P}_2 \oplus \mathbf{P}_3 \qquad \mathsf{E} \otimes \mathbf{P}_i = \mathbf{P}_i \otimes \mathsf{E} = \mathsf{E}$$

*Confirmed $\mathsf{Rep}(D_8)$ over other $\mathsf{TY}(D_4)$ via gauging

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A curious SPT

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$U^2 = 1,$ ... $R_i R_{\mathsf{E}} = R_{\mathsf{E}}$

H is in a $\mathbb{Z}_2 \times \mathsf{Rep}(D_8)$ weak SPT phase

- Translation defects carry $\mathsf{Rep}(D_8)$ symmetry charge in $|\mathrm{GS}\rangle$

➤ Form a ...

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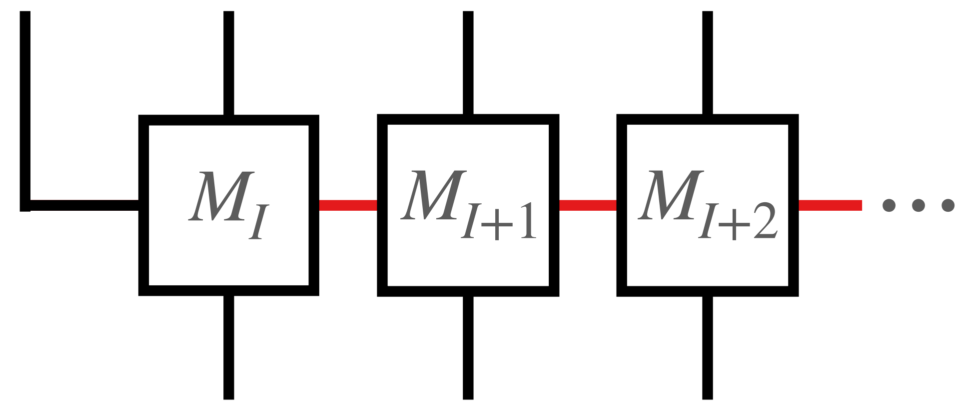
Inserting an R_{E} symmetry defect

An R_{E} symmetry defect can be inserted using the MPO presentation of R_{E}

$$R_{\mathsf{E}}^{(I)} =$$



- Maps states in $\mathscr{H} \cong \mathbb{C}^{8L}$ to those in $\mathscr{H}_{\mathsf{E}} \cong \mathbb{C}^2 \otimes \mathscr{H}$

Defect Hamiltonian $(R_{\mathsf{E}}^{(I)} H = H_{\mathsf{E}}^{(I-1,I)} R_{\mathsf{E}}^{(I)})$

$$H_{\mathsf{E}}^{(I-1,I)} = H + (1 - Z_{\text{defect}})\, \sigma_{I-1}^{x} C_I \sigma_I^{x}$$

- Two exactly degenerate ground states

$$|\mathrm{GS}_+\rangle = |+1\rangle \otimes |\mathrm{GS}\rangle \qquad |\mathrm{GS}_-\rangle = |-1\rangle \otimes |\widetilde{\mathrm{GS}}\rangle$$

Inserting an R_{E} symmetry defect

An R_{E} symmetry defect can be inserted using the MPO

E-twisted symmetry operators satisfy

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➤ Maps states in … to those in …

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A curious projective algebra

This SPT is characterized by a projective symmetry:

$$U R_{\mathsf{E}} = -R_{\mathsf{E}} U \qquad \text{(odd } L\text{)}$$

Projective unitary symmetries $U_1 U_2 = \mathrm{e}^{\mathrm{i}\theta} U_2 U_1$ forbid SPTs

➤ Assume non-degenerate symmetric ground state $|\mathrm{GS}\rangle$

$$\left.\begin{array}{l} 1.\ \ U_1 U_2 |\mathrm{GS}\rangle = |\mathrm{GS}\rangle \\ 2.\ \ U_1 U_2 |\mathrm{GS}\rangle = \mathrm{e}^{\mathrm{i}\theta} U_2 U_1 |\mathrm{GS}\rangle = \mathrm{e}^{\mathrm{i}\theta} |\mathrm{GS}\rangle \end{array}\right\} \textit{Contradicts assumption}$$

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2. $U_1 U_2 |\mathrm{GS}\rangle = \mathrm{e}^{\mathrm{i}\theta} U_2 U_1 |\mathrm{GS}\rangle = \mathrm{e}^{\mathrm{i}\theta} |\mathrm{GS}\rangle$

} *Contradicts assumption*

Projective non-invertible symmetries <u>are compatible</u> with SPTs

➤ Loophole: symmetry operator has zero-eigenvalues

$$U R_{\mathsf{E}} = (-1)^L R_{\mathsf{E}} U \implies R_{\mathsf{E}} |\mathrm{SPT}\rangle = 0 \text{ when } L \text{ is odd}$$

Non-invertible symmetry and SPTs

SPTs protected by internal invertible versus non-invertible symmetry [Thorngren, Wang '19; Inamura '21; Fechisin, Tantivasadakarn, Albert '23; Antinucci, Bhardwaj, Bottini, Copetti, Gai, Huang, Pajer, Schäfer-Nameki, Tiwari, Warman, Wu '23-25; Seifnashri, Shao '24; Li, Litvinov '24; Jia '24; Inamura, Ohyama '24; Meng, Yang, Lan, Gu '24; Cao, Yamazaki, Li '25; Aksoy, Wen '25]

Properties	Invertible	Non-invertible
Stacking/Entanglers	Yes	No
Classification	Cobordism	Fiber functors
Edge/interface modes	Yes	Yes
Defect characterization	Yes	Yes
Compatible with projectivite algebras	No	Yes

Projective $Z(G) \times \mathsf{Rep}(G)$ symmetry

The projective $\mathbb{Z}_2 \times \mathsf{Rep}(D_8)$ symmetry is a special case of a projective $Z(G) \times \mathsf{Rep}(G)$ symmetry

- $Z(G)$ is the center of a finite group G
- $\mathsf{Rep}(G)$ is the fusion category of representations of G

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Onsite $Z(G)$ symmetry operator $U_z = \prod_j U_j^{(z)}$, with $z \in Z(G)$:

$$U_{z_1} U_{z_2} = U_{z_1 z_2}$$

$\mathsf{Rep}(G)$ symmetry operator R_Γ, with Γ an irrep of G:

$$R_{\Gamma_a} \times R_{\Gamma_b} = \sum_c N_{ab}^c R_{\Gamma_c}$$

- Non-invertible symmetry when G is non-Abelian

Projective $Z(G) \times \mathsf{Rep}(G)$ symmetry

The projectivity arises through the local relation

$$R_\Gamma U_j^{(z)} = \mathrm{e}^{\mathrm{i}\phi_\Gamma(z)}\, U_j^{(z)} R_\Gamma \text{ with } \mathrm{e}^{\mathrm{i}\phi_\Gamma(z)} = \mathrm{Tr}[\Gamma(z)]\,/\,d_\Gamma$$

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e.g., $\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)}$ when $G = \mathbb{Z}_2$ $(Z(\mathbb{Z}_2) = \mathbb{Z}_2)$

z \ Γ	**1**	sign
$+1$	$+1$	$+1$
-1	$+1$	-1

- The symmetries of XY model we saw in morning talk

$$R_{\mathrm{sign}} = \prod_{j=1}^{L} Z_j \qquad\qquad U_{-1} = \prod_{j=1}^{L} X_j$$

Projective $Z(G) \times \mathsf{Rep}(G)$ symmetry

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e.g., $\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)}$ when $G = D_8$ $(Z(D_8) = \mathbb{Z}_2)$

$z \backslash \Gamma$	$\mathbf{1}$	$\mathbf{1}_1$	$\mathbf{1}_2$	$\mathbf{1}_3$	E
$+1$	$+1$	$+1$	$+1$	$+1$	$+1$
-1	$+1$	$+1$	$+1$	$+1$	-1

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$+1$	$+1$	$+1$	$+1$	$+1$	$+1$
-1	$+1$	$+1$	$+1$	$+1$	-1

Explicit expressions of U_z and R_Γ for the Hilbert space $\bigotimes_j \mathbb{C}^{|G|}$

$$U_z = \sum_{\{g_j\}} |zg_1, \cdots, zg_L\rangle\langle g_1, \cdots, g_L| \qquad R_\Gamma = \sum_{\{g_j\}} \mathrm{Tr}[\Gamma(g_1 \cdots g_L)] \, |g_1, \cdots, g_L\rangle\langle g_1, \cdots, g_L|$$

Constraints from projectivity

The local projective algebra implies $R_\Gamma U_z = (\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)})^L U_z R_\Gamma$

- When $\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)}$ is non-trivial for a unitary R_Γ, this is a manifestation of a Lieb-Schultz-Mattis (LSM) anomaly
- The LSM anomaly forbids SPT phases

  [Lieb, Schultz, Mattis '61; Oshikawa '99; Hastings '03; ⋯; Chen, Gu, Wen '10; Else, Thorngren '19; Yao, Oshikawa '20; Ogata, Tasaki '21; Cheng, Seiberg '22; Seifnashri '23; Kapustin, Sopenko '24]

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When $\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)}$ is non-trivial for only non-invertible R_Γ, there is the $R_\Gamma|\mathrm{SPT}\rangle = 0$ loophole $\Longrightarrow$ Can have an SPT,

- Does this projective algebra then have any consequences?

Yes! There is an SPT-LSM theorem

SPT-LSM theorems

An SPT-LSM theorem is an obstruction to a trivial SPT*

[Lu '17; Yang, Jiang, Vishwanath, Ran '17; Lu, Ran, Oshikawa '17; Else, Thorngren '19; Jiang, Cheng, Qi, Lu '19]

- Any SPT state must have non-zero entanglement

Symmetry-enforced entanglement

*Trivial SPT = symmetric product state, which is a non-canonical choice

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Symmetry-enforced entanglement

Why does the projective algebra

$$R_\Gamma U_z = (\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)})^L \, U_z R_\Gamma$$

gives rise to an SPT-LSM theorem?

- Local projective algebra forbids a trivial SPT
- Any $|\mathrm{SPT}\rangle$ must satisfy $R_\Gamma |\mathrm{SPT}\rangle = 0$ when $(\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)})^L \neq 1$

*Trivial SPT = symmetric product state, which is a non-canonical choice

Simple SPT-LSM example

Consider a $1+1$D system with two $\mathbb{Z}_4$ qudits on each site $j \sim j+L$ with L even and $\mathbb{Z}_4 \times \mathbb{Z}_4$ symmetry operators

$$U = \prod_j X_j \tilde{X}_j \qquad\qquad V = \prod_j (Z_j \tilde{Z}_j)^{2j+1}$$

Local projective algebra $U_j V_j = -V_j U_j$, but no LSM anomaly

[Jiang, Cheng, Qi, Lu '19]

➤ Defect perspective: Inserting a U symmetry defect causes

$$T_{\text{tw}} V = \Big(-\prod Z_j^2 \tilde{Z}_j^2\Big) V T_{\text{tw}}$$

Non-abelian group, not a projective rep!

Furthermore, there is no trivial $|\text{SPT}\rangle = \bigotimes_j |\psi_j\rangle$

➤ Easily proven by contradiction using $U_j V_j = -V_j U_j$

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Local projective algebra $U_j V_j = -V_j U_j$, but no LSM anomaly [Jiang, Cheng, Qi, Lu '19]

➤ Defec ... causes

...lian group, ...jective rep!

Key point: obstruction to product state SPT while keeping U and V onsite

➤ Common for modulated SPTs [SP, *work in progress*]

Furthermore, there is no trivial $|\mathrm{SPT}\rangle = \bigotimes_j |\phi_j\rangle$

➤ Easily proven by contradiction using $U_j V_j = -V_j U_j$

SPT-LSM theorem proof

To prove our SPT-LSM theorem, we

1. Use that the $Z(G)$ symmetry is on-site:

$$U_z = \prod_j U_j^{(z)} \quad \text{which satisfies} \quad R_\Gamma U_j^{(z)} = \mathrm{e}^{\mathrm{i}\phi_\Gamma(z)}\, U_j^{(z)} R_\Gamma$$

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2. Use that any translation-inv product state $|\mathrm{GS}\rangle$ satisfies $R_\Gamma|\mathrm{GS}\rangle \neq 0$ for some $L = L^* \left(\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)L^*} = 1\right)$

   $= d_\Gamma$ for $L = |G|\mathbb{Z}$

   - For $\mathscr{H}_j = \mathbb{C}^{|G|}$, $R_\Gamma \bigotimes_{j=1}^{L} \sum_{g\in G} c_g|g\rangle = \chi_\Gamma(\tilde{g}^L) c_{\tilde{g}}^L |\tilde{g}\cdots\tilde{g}\rangle + \cdots$

   - Generally true if there is an IR TQFT description since

$$R_\Gamma|\mathrm{GS}_{\mathrm{TQFT}}\rangle = d_\Gamma|\mathrm{GS}_{\mathrm{TQFT}}\rangle$$

SPT-LSM theorem proof

If there is an SPT state $|\mathrm{GS}\rangle$ that is a product state:

- $U_z|\mathrm{GS}\rangle = |\mathrm{GS}\rangle \Longrightarrow U_j^{(z)}|\mathrm{GS}\rangle = |\mathrm{GS}\rangle$

Using that $R_\Gamma|\mathrm{GS}\rangle = \lambda_\Gamma|\mathrm{GS}\rangle \neq 0$ at $L = L^*$:

1. $R_\Gamma U_j^{(z)}|\mathrm{GS}\rangle = R_\Gamma|\mathrm{GS}\rangle = \lambda_\Gamma|\mathrm{GS}\rangle$ ← *Contradiction*
2. $R_\Gamma U_j^{(z)}|\mathrm{GS}\rangle = \mathrm{e}^{\mathrm{i}\phi_\Gamma(z)} U_j^{(z)} R_\Gamma|\mathrm{GS}\rangle = \lambda_\Gamma \mathrm{e}^{\mathrm{i}\phi_\Gamma(z)}|\mathrm{GS}\rangle$

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$\implies$ Cannot be an SPT state that is a product state at $L = L^*$

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$\implies$ Cannot be an SPT state that is a product state at $L = L^*$

$\implies$ By locality, there cannot be an SPT state that is a product state for any L

SPT-LSM theorem proof

If there is an SPT state $|\mathrm{GS}\rangle$ that is a product state:

➤ $U_z|\mathrm{GS}\rangle = |\mathrm{GS}\rangle \implies U_j^{(z)}|\mathrm{GS}\rangle = |\mathrm{GS}\rangle$

Usin…

1. R … *ion*

2. R …

Therefore, the projective non-invertible symmetry prevents a product state SPT

➤ All SPTs must have non-zero entanglement

$\implies$ Cannot be an SPT state that is a product state at $L = L^*$

$\implies$ By locality, there cannot be an SPT state that is a product state for any L

Non-invertible weak SPT

What is the characterization of these SPTs?

- They must satisfy $R_\Gamma |\mathrm{GS}\rangle = 0$ for nontrivial $(\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)})^L$

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Two possibilities:

1. An SPT state satisfies $R_\Gamma|\mathrm{GS}\rangle = 0$ for all system sizes L
2. For $L = L^*$ where all $(\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)})^{L^*} = 1$, an SPT state satisfies $R_\Gamma|\mathrm{GS}\rangle = \lambda_\Gamma|\mathrm{GS}\rangle \neq 0$, but $R_\Gamma|\mathrm{GS}\rangle = 0$ for $L \neq L^*$

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The first is incompatible with an IR TQFT

Non-invertible weak SPT

What is the characterization of these SPTs?

At $L = L^*$, SPTs satisfy $R_\Gamma |\mathrm{GS}\rangle = \lambda_\Gamma |\mathrm{GS}\rangle \neq 0$

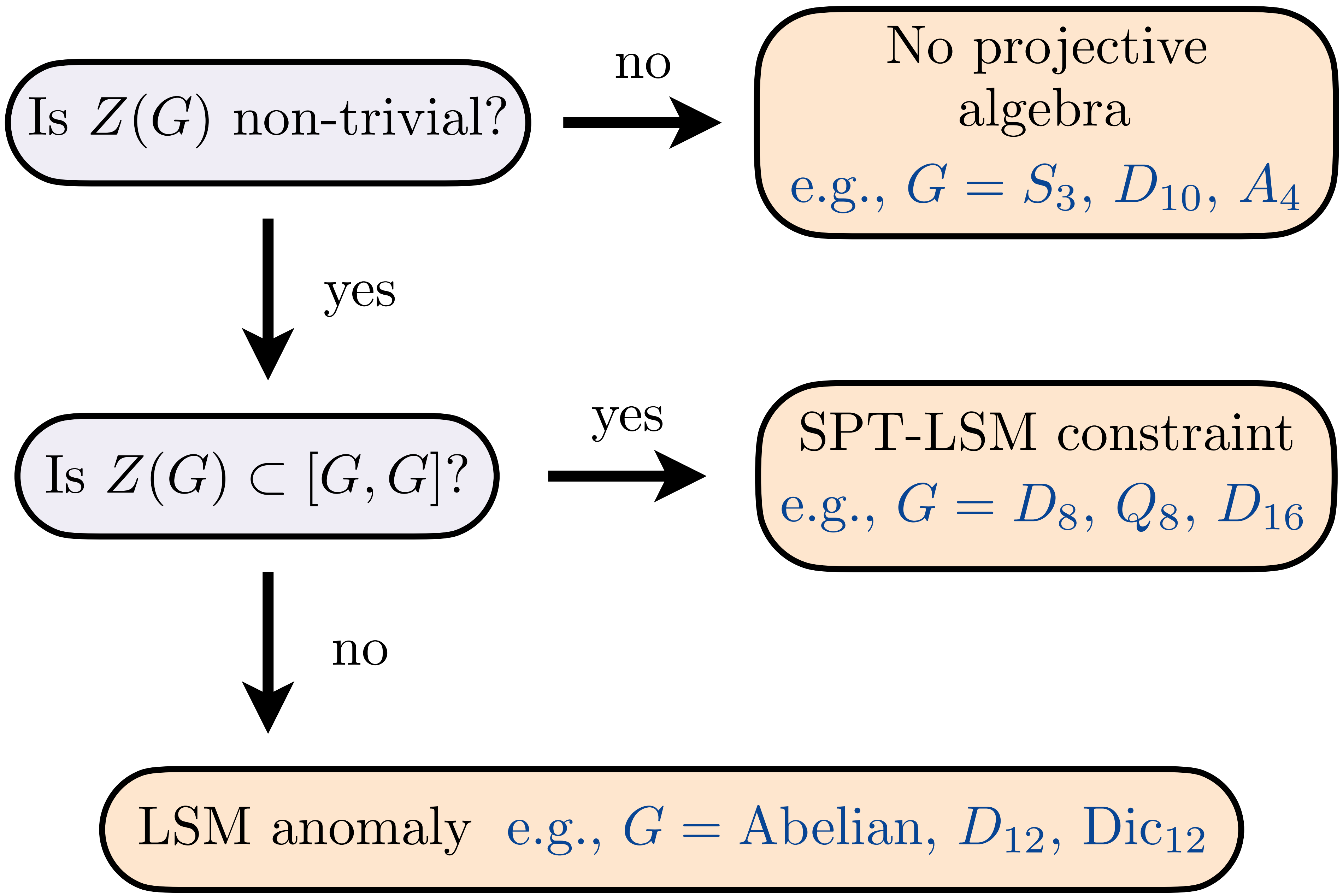
At $L = L^* + 1$, SPTs satisfy $R_\Gamma |\mathrm{GS}\rangle = 0$

- All SPT states have translation defects dressed by non-trivial $\mathsf{Rep}(G)$ symmetry charge
- $\nexists$ a trivial SPT $\Longrightarrow$ SPT-LSM theorem

The first is incompatible with an IR TQFT

(SPT)-LSM theorems

Whether there is an (SPT)-LSM theorem depends on G:



Outlook

We found a new class of entangled weak SPTs characterized by a projective $Z(G) \times \mathsf{Rep}(G)$ non-invertible symmetry

1. An exactly solvable model in a weak SPT phase characterized by a projective $\mathbb{Z}_2 \times \mathsf{Rep}(D_8)$ symmetry

2. General discussion on projective $Z(G) \times \mathsf{Rep}(G)$ weak SPTs $\implies$ an SPT-LSM theorem

For the newcomer: New quantum phases and models can be discovered using generalized symmetries as a guide!

For the initiated: Beyond-relativistic-QFT-symmetries are interesting!

Back-up slides

The surprising lack of an ’t Hooft anomaly

Inserting U or R_{E} symmetry defects leads to the projective algebras

U *symmetry defect*	R_{E} *symmetry defect*
$R_{\mathsf{E}}T = -TR_{\mathsf{E}}$	$TU = -UT$

For invertible symmetries, such projective algebras imply an ’t Hooft anomaly (e.g., the type III anomaly $(-1)^{\int_{M_3} a\cup b\cup c}$)

[Matsui ’08; Yao, Oshikawa ’20; Seifnashri ’23; Kapustin, Sopenko ’24]

➤ This is not true for non-invertible symmetries!

The surprising lack of an 't Hooft anomaly

Inserting U or R_{E} symmetry defects leads to the projective algebras

U *symmetry defect*	R_{E} *symmetry defect*
$R_{\mathsf{E}}\,T = -\,T\,R_{\mathsf{E}}$	$T\,U = -\,U\,T$

Fails because of $R_{\mathsf{E}} = 0$ loophole

For invertib… …metries, such projective algebras imply an 't … the type III anomaly $(-1)^{\int_{M_3} a\cup b\cup c}$)

…; Kapustin, Sopenko '24]

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For invertib... metries, such projective ...bras imply an ’t ... the type ... Kapustin, Sope...

➤ This is not true for non-inve...

Fails because of $R_\mathsf{E} = 0$ loophole

Fails because the degeneracy is encoded in the defect’s quantum dimension

Projective $\mathbb{Z}_2 \times \mathsf{Rep}(D_8)$ bond algebra

$$\mathfrak{B}\left[\mathsf{Rep}(D_8) \times \mathbb{Z}_2\right] = \left\langle \sigma_j^z,\ Z_j^2,\ Z_j\, Z_{j+1},\ \sigma_j^x\, C_{j+1}\, \sigma_{j+1}^x,\ X_j^{\sigma_j^z}\, X_{j+1}^\dagger \right\rangle$$

Example: $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPTs

1d closed chain in space with two qubits on each site $j \sim j + L$ acted on by Pauli operators X_j, Z_j and $\tilde{X}_j$, $\tilde{Z}_j$.

$$H_p = -\sum_{j=1}^{L} (X_j + \tilde{X}_j)$$

$$|\mathrm{GS}_p\rangle = |+ + \cdots +\rangle$$

$$H_c = -\sum_{j=1}^{L} \left(\tilde{Z}_{j-1} X_j \tilde{Z}_j + Z_j \tilde{X}_j Z_{j+1}\right)$$

$$|\mathrm{GS}_c\rangle = \tilde{Z}_{j-1} X_j \tilde{Z}_j |\mathrm{GS}_c\rangle = Z_j \tilde{X}_j Z_{j+1} |\mathrm{GS}_c\rangle$$

- Both models have a unique symmetric gapped ground state
- There is a $\mathbb{Z}_2 \times \tilde{\mathbb{Z}}_2$ symmetry $U = \prod_j X_j$ and $\tilde{U} = \prod_j \tilde{X}_j$ with $U|\mathrm{GS}_\bullet\rangle = \tilde{U}|\mathrm{GS}_\bullet\rangle = |\mathrm{GS}_\bullet\rangle$

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H_p and H_c are both in a $\mathbb{Z}_2 \times \tilde{\mathbb{Z}}_2$ SPT phase

- There is a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry $U = \prod_j X_j$ and $\tilde{U} = \prod_j \tilde{X}_j$ with $U|\mathrm{GS}_\bullet\rangle = \tilde{U}|\mathrm{GS}_\bullet\rangle = |\mathrm{GS}_\bullet\rangle$

Distinguishing $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPTs

Are H_p and H_c in different $\mathbb{Z}_2 \times \tilde{\mathbb{Z}}_2$ SPT phases?

We can check by inserting a U symmetry defect at $\langle L, 1 \rangle$

➤ Gives rise to U-twisted boundary conditions: $Z_{j+L} = -Z_j$

1. H_p is unaffected, so its ground state still satisfies

$$U\,|\mathrm{GS}_{p;U}\rangle = +\,|\mathrm{GS}_{p;U}\rangle \qquad \tilde{U}\,|\mathrm{GS}_{p;U}\rangle = +\,|\mathrm{GS}_{p;U}\rangle$$

2. H_c becomes $H_c + 2\,Z_L\,\tilde{X}_L\,Z_1$, and its ground state satisfies

$$U\,|\mathrm{GS}_{c;U}\rangle = +\,|\mathrm{GS}_{c;U}\rangle \qquad \tilde{U}\,|\mathrm{GS}_{c;U}\rangle = -\,|\mathrm{GS}_{c;U}\rangle$$

Distinguishing $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPTs

We

- C

Different responses imply that H_p and H_c are in different $\mathbb{Z}_2 \times \tilde{\mathbb{Z}}_2$ SPT phases

[Chen, Lu, Vishwanath 2013; Gaiotto, Johnson-Freyd 2017; Wang, Ning, Cheng 2021]

Low-energy EFTs of H_p and H_c

$$Z_p[A, \tilde{A}] = 1 \qquad Z_c[A, \tilde{A}] = (-1)^{\int A \cup \tilde{A}}$$

1. H_p is unaffected, so its ground state still satisfies

$$U\,|\mathrm{GS}_{p;U}\rangle = +\,|\mathrm{GS}_{p;U}\rangle \qquad \tilde{U}\,|\mathrm{GS}_{p;U}\rangle = +\,|\mathrm{GS}_{p;U}\rangle$$

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$$U\,|\mathrm{GS}_{c;U}\rangle = +\,|\mathrm{GS}_{c;U}\rangle \qquad \tilde{U}\,|\mathrm{GS}_{c;U}\rangle = -\,|\mathrm{GS}_{c;U}\rangle$$

LSM anomaly in the XY model

Many-qubit model on a periodic chain with Hamiltonian

$$H = \sum_{j=1}^{L} J\,\sigma_j^x\,\sigma_{j+1}^x + K\,\sigma_j^y\,\sigma_{j+1}^y$$

- There is an LSM anomaly involving the $\mathbb{Z}_2^x \times \mathbb{Z}_2^y \times \mathbb{Z}_L$ symmetry [Chen, Gu, Wen 2010; Ogata, Tasaki 2021]

$$U_x = \prod_j \sigma_j^x, \quad U_y = \prod_j \sigma_j^y, \quad \text{and lattice translations } T$$

- Manifests through the projective algebras [Cheng, Seiberg 2023]

Translation defects	$\mathbb{Z}_2^x$ *defect*	$\mathbb{Z}_2^y$ *defect*
$U_x U_y = (-1)^L U_y U_x$	$U_y T = -T U_y$	$T U_x = -U_x T$

GROUP BASED QUDITS

A G-qudit is a $|G|$-level quantum mechanical system whose states are $|g\rangle$ with $g \in G$

- G is a finite group, e.g. $\mathbb{Z}_2$, S_3, D_8, SmallGroup(32,49)

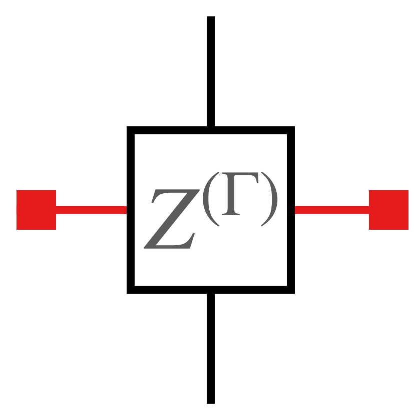
Group based Pauli operators [Brell 2014]

- X operators labeled by group elements

$\bar{g} \equiv g^{-1}$

$$\overrightarrow{X}^{(g)} = \sum_h |gh\rangle\langle h| \qquad \overleftarrow{X}^{(g)} = \sum_h |h\bar{g}\rangle\langle h|$$

- Z operators are MPOs labeled by irreps $\Gamma: G \to \mathrm{GL}(d_\Gamma, \mathbb{C})$

$$[Z^{(\Gamma)}]_{\alpha\beta} = \sum_h [\Gamma(h)]_{\alpha\beta} |h\rangle\langle h| \equiv \alpha$$ 

$\beta \qquad (\alpha, \beta = 1,2,\cdots, d_\Gamma)$

GROUP BASED QUDITS

Example: $G = \mathbb{Z}_2$ where $g \in \{1, -1\}$ and $\Gamma \in \{\mathbf{1}, \mathbf{1}'\}$

$$\overrightarrow{X}^{(1)} = \overleftarrow{X}^{(1)} = [Z^{(\mathbf{1})}]_{11} = 1$$

$$\overrightarrow{X}^{(-1)} = \overleftarrow{X}^{(-1)} = \sigma^x \qquad [Z^{(\mathbf{1}')}]_{11} = \sigma^z$$

Group based Pauli operators satisfy

1. $\overrightarrow{X}^{(g)}\overrightarrow{X}^{(h)} = \overrightarrow{X}^{(gh)}$, $\overleftarrow{X}^{(g)}\overleftarrow{X}^{(h)} = \overleftarrow{X}^{(gh)}$, and $\overrightarrow{X}^{(g)}\overleftarrow{X}^{(h)} = \overleftarrow{X}^{(h)}\overrightarrow{X}^{(g)}$
2. $\overrightarrow{X}^{(g)}\overrightarrow{X}^{(h)} = \overrightarrow{X}^{(h)}\overrightarrow{X}^{(g)}$ iff g and h commute
3. $\overrightarrow{X}^{(g)}[Z^{(\Gamma)}]_{\alpha\beta} = [\Gamma(\bar{g})]_{\alpha\gamma}[Z^{(\Gamma)}]_{\gamma\beta}\overrightarrow{X}^{(g)}$
4. Unitarity: $\overrightarrow{X}^{(g)\dagger} = \overrightarrow{X}^{(\bar{g})}$, $\overleftarrow{X}^{(g)\dagger} = \overleftarrow{X}^{(\bar{g})}$, $[Z^{(\Gamma)\dagger}Z^{(\Gamma)}]_{\alpha\beta} = \delta_{\alpha\beta}$

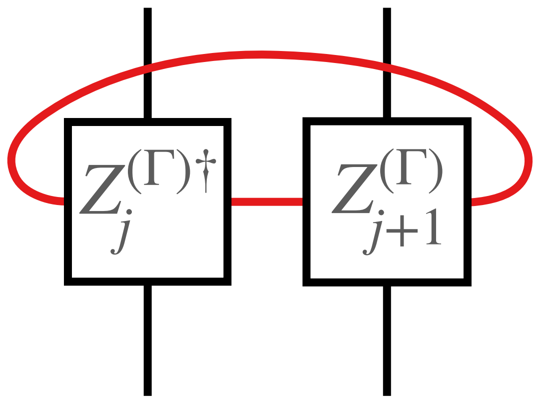
GROUP BASED XY MODEL

Group based Pauli operators are useful for constructing quantum lattice models [Brell 2014; Albert *et. al.* 2021; Fechisin, Tantivasadakarn, Albert 2023]

Group based XY model: Consider a periodic 1d lattice of L sites. On each site j resides a G-qudit and its Hamiltonian

$$H_{XY} = \sum_{j=1}^{L} \left(\sum_{\Gamma} J_\Gamma \operatorname{Tr}\left(Z_j^{(\Gamma)\dagger} Z_{j+1}^{(\Gamma)} \right) + \sum_g K_g \overleftarrow{X}_j^{(g)} \overrightarrow{X}_{j+1}^{(g)} \right) + \mathrm{hc}$$

$$\operatorname{Tr}\left(Z_j^{(\Gamma)\dagger} Z_{j+1}^{(\Gamma)} \right) = \sum_{\{g\}} \chi_\Gamma(\bar{g}_j g_{j+1}) \, | \{g\} \rangle \langle \{g\} | \equiv$$



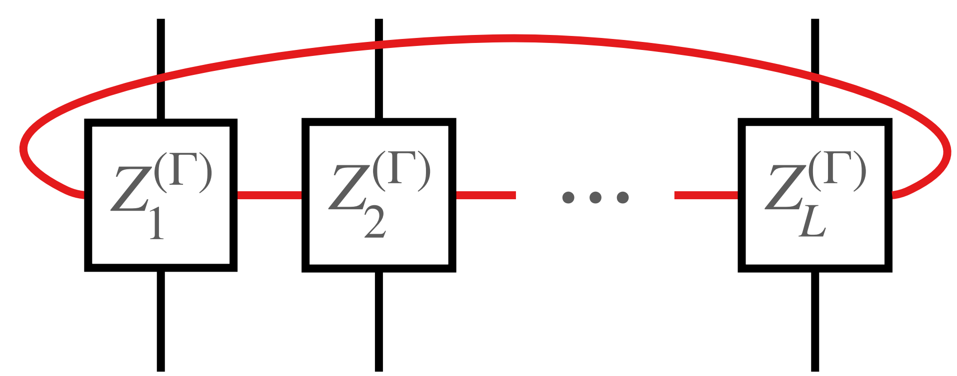
- For $G = \mathbb{Z}_2$, this is the ordinary quantum XY model

SYMMETRY OPERATORS

$$H_{XY} = \sum_{j=1}^{L} \left(\sum_{\Gamma} J_\Gamma \, \mathrm{Tr}\left(Z_j^{(\Gamma)\dagger} Z_{j+1}^{(\Gamma)} \right) + \sum_g K_g \, \overleftarrow{X}_j^{(g)} \, \overrightarrow{X}_{j+1}^{(g)} \right) + \mathrm{hc}$$

$\mathbb{Z}_L$ lattice translations: $T \mathcal{O}_j T^\dagger = \mathcal{O}_{j+1}$

Various internal symmetries:

- $Z(G)$ symmetry $U_z = \prod_j \overrightarrow{X}_j^{(z)}$ with $z \in Z(G)$
- Rep(G) symmetry $R_\Gamma = \mathrm{Tr}\left(\prod_{j=1}^{L} Z_j^{(\Gamma)} \right) \equiv$ 

$$R_{\Gamma_a} \times R_{\Gamma_b} = R_{\Gamma_a \otimes \Gamma_b} = R_{\oplus_c N_{ab}^c \Gamma_c} = \sum_c N_{ab}^c R_{\Gamma_c}$$

PROJECTIVE ALGEBRA FROM DEFECTS

$$U_z = \prod_j \vec{X}_j^{(z)} \qquad R_\Gamma = \mathrm{Tr}\left(\prod_{j=1}^{L} Z_j^{(\Gamma)}\right)$$

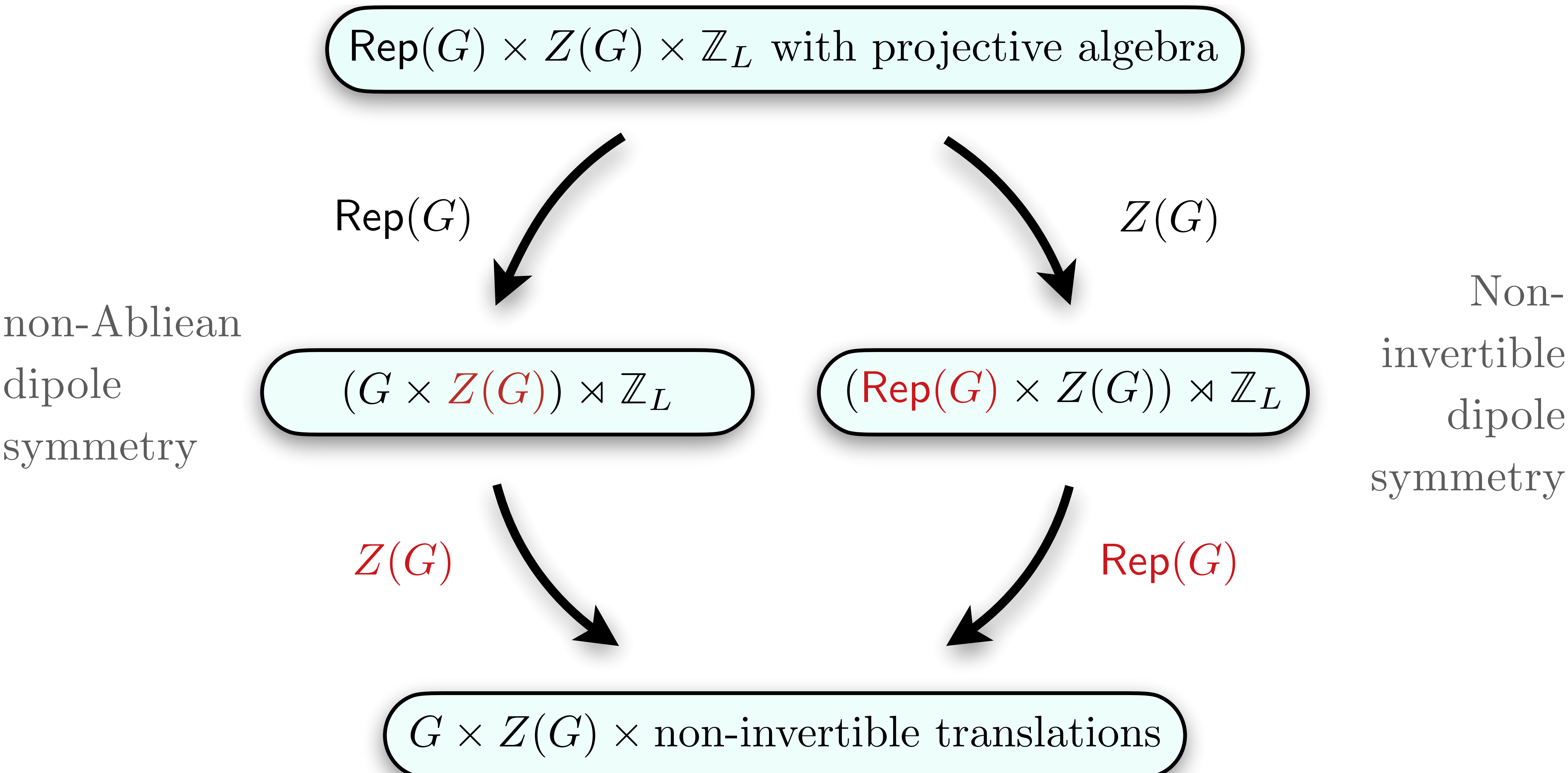
$$T_{\mathrm{tw}}^{(z)} = \vec{X}_I^{(z)}\, T \qquad T_{\mathrm{tw}}^{(\Gamma)} = \widehat{Z}_I^{(\Gamma)}\,(T \otimes \mathbf{1})$$

Letting $e^{i\phi_\Gamma(z)} \equiv \chi_\Gamma(z)/d_\Gamma$

Translation defects	$z \in Z(G)$ *defect*	$\Gamma \in \mathsf{Rep}(G)$ *defect*
$R_\Gamma U_z = (e^{i\phi_\Gamma(z)})^L\, U_z R_\Gamma$	$R_\Gamma T_{\mathrm{tw}}^{(z)} = e^{i\phi_\Gamma(z)} T_{\mathrm{tw}}^{(z)} R_\Gamma$	$T_{\mathrm{tw}}^{(\Gamma)}\, U_z = e^{i\phi_\Gamma(z)} U_z\, T_{\mathrm{tw}}^{(\Gamma)}$

- Generalizes the $G = \mathbb{Z}_2$ projective algebra of the ordinary quantum XY model

GAUGING WEB


$\mathsf{Rep}(G) \times Z(G) \times \mathbb{Z}_L$ with projective algebra
$\mathsf{Rep}(G)$
$Z(G)$
non-Abliean dipole symmetry
$(G \times Z(G)) \rtimes \mathbb{Z}_L$
$(\mathsf{Rep}(G) \times Z(G)) \rtimes \mathbb{Z}_L$
Non-invertible dipole symmetry
$Z(G)$
$\mathsf{Rep}(G)$
$G \times Z(G) \times$ non-invertible translations