

Entangled weak SPTs from projective non-invertible symmetries

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OIST Generalized Symmetries in Quantum Matter Thematic Program





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SP, Lam, Aksoy arXiv:2409.18113
[SciPost Phys. 18, 028 (2025)]

Quantum phases and symmetry

A fundamental problem in **CMT/QFT/Math-ph** is to understand **quantum phases**

1. How do we diagnose different quantum phases?
2. What are the allowed possible quantum phases?

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Sometimes, phases are characterized by a **symmetry**

- **Superfluids** by $U(1)$ boson number conservation
- **Topological insulators** by $U(1)_f$ and time-reversal

For such **phases, symmetries** provide answers to questions (1) and (2).

Generalized symmetries

There has been a recent flurry of interest in **generalizing** the notion of **symmetries**

- **Ordinary** symmetries transform **local** operators in an **invertible** manner (e.g., $c_r^\dagger \rightarrow e^{i\theta} c_r^\dagger$)
- So-called **generalized symmetries** modify this definition

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Non-invertible symmetries have non-invertible transformations

[Bhardwaj, Tachikawa '17; Chang, Lin, Shao, Wang, Yin '18; ...]

- Can arise at **critical points** from Kramers-Wannier dualities
[Thorngren, Yang '21 ; Choi, Córdova, Hsin, Lam, Shao '21; ...]
- Can emerge in **ordered phases** (are symmetries of nonlinear sigma models) [Chen, Tanizaki '22; Hsin '22; SP '23; SP, Zhu, Beaudry, X-G Wen '23]

Generalized symmetries

Q: Why should we consider these as **symmetries**?

The
no



No

[Bha



Sig

Generalized symmetries

Q: Why should we consider these as **symmetries**?

A: They pass the **duck test**!



If it looks like a duck, swims like a duck, and quacks like a duck, then it probably is a duck.

- Have conservation laws
- Can constrain phase diagrams (be anomalous)
- Can characterize SSB and SPT phases

Quantum phases + generalized symmetry

Which quantum phases are characterized by
generalized symmetries?

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Systematic build-a-phase recipe

(1) Choose your generalized symmetries adjectives

$a_1 - a_2 - a_3 - \dots$ Symmetry

(2) Specify “SSB and SPT pattern”

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Ordered phases

Topological insulators

Topological order

Maxwell phases

Higgs phases

Fracton phases

Phases we have yet to name!

Quantum phases + generalized symmetry

Which quantum phases are characterized by
generalized symmetries?

Why care?

1. Provides a **novel** and **unifying** perspective of **quantum phases**
2. Guides us towards new **quantum phases** and models
3. Further develops a classification of **quantum phases** based on **symmetries** (a “generalized Landau paradigm”)

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Quantum phases + generalized symmetry

Which quantum phases are characterized by
generalized symmetries?

Wh

We are making incredible progress!

1. [Albert, Aksoy, Atinucci, Barkeshli, Bhardwaj, Bottini, Bulmash, Burnell, Cao, Chatterjee, Chen, Cheng, Choi, Copetti, Córdova, Delcamp, Delfino, Devakul, Dua, Dumitrescu, Eck, Fechis in, Fendley, Gai, Gaiotto, Garre-Rubio, Gorantla, Gu, Han, Hsin, Huang, Inamura, Ji, Jia, Jian, Kapustin, Kobayashi, Kong, Lake, Lam, Lan, Lee, Li, Litvinov, Liu, Lootens, Ma, Meng, Molnár, Myerson-Jain, Nandkishore, Oh, Ohmori, Pajer, Pichler, Prem, Rayhaun, Sanghavi, Schäfer-Nameki, Seiberg, Seifnashri, Shao, Sondhi, Stahl, Stephen, Tantivasadakarn, Thorngren, Tiwari, Tsui, Ueda, Verresen, Verstraete, Vijay, Wang, Warman, Wen, Willet, Williamson, Wu, Xu, Yamazaki, Yan, Yang, Yoshida, Zhang, Zheng, ...]
- 2.
3. Here: focus on beyond-relativistic-QFT-symmetries

On symmetries (a generalized Landau paradigm)

TL;DR for this talk

This talk: 1 + 1D SPT phases characterized by lattice translation and **non-invertible symmetries**

- Find a new class of entangled weak SPTs characterized by **projective non-invertible symmetries** on the lattice

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Outline

1. Weak SPTs from a **symmetry defect** perspective
2. Simple example of an entangled weak SPT characterized by a **projective non-invertible symmetry**
3. General discussion on **projective $Z(G) \times \text{Rep}(G)$ symmetry** and **(SPT-)LSM theorems**

Recap: SPTs and symmetry defects

Recall: An SPT phase is a gapped quantum phase protected by a **symmetry** with a **unique ground state** on all closed spatial manifolds [Chen, Gu, Liu, Wen 2011; ...]

- SPTs are characterized by their bulk **response** to static **probes**: Background gauge fields and **symmetry defects**

Recap: SPTs and symmetry defects

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- SPTs are characterized by their bulk **response to static probes**: Background gauge fields and **symmetry defects**

Recall: **Symmetry defects** are localized modifications to the Hamiltonian $H_{\text{defect}}^{(\Sigma)} = H + \delta H(\Sigma)$ and other operators

- Moved using **unitary operators** (are topological defects)
- **Twisted** boundary conditions $(T_{\perp})^L = \text{Symmetry operator}$

Example: \mathbb{Z}_2 weak SPTs

SPTs can be protected by internal and **spacetime** symmetries

- SPTs protected by $G \times$ **translations** are called **weak G-SPTs**

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Example: 1d periodic lattice with a **qubit** on each site $j \sim j + L$

$$H_+ = - \sum_j X_j \quad \text{vs.} \quad H_- = + \sum_j X_j$$

- Both have a unique gapped ground state $|\text{GS}_\pm\rangle = \otimes_j |\pm\rangle$
- **Symmetries:** $\mathbb{Z}_2 \times \mathbb{Z}_L$ with $U = \prod_j X_j$ and $T: j \rightarrow j + 1$

H_+ and H_- are both in \mathbb{Z}_2 weak SPT phases

Example: \mathbb{Z}_2 weak SPTs

Are H_+ and H_- in different \mathbb{Z}_2 weak SPT phases?

Let's insert a $U = \prod_j X_j$ symmetry defect at $\langle L, 1 \rangle$

- Neither H_+ or H_- are modified by $Z_{j+L} = -Z_j$
- Translation operator becomes $T = X_1 T_{\text{defect-free}}$ ($T^L = U$)

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| | Even L | Even L , \mathbb{Z}_2 symmetry defect |
|-------------------------------|-----------------------------|--|
| $U \text{GS}_\pm \rangle =$ | $+ \text{GS}_\pm \rangle$ | $+ \text{GS}_\pm \rangle$ |
| $T \text{GS}_\pm \rangle =$ | $+ \text{GS}_\pm \rangle$ | $\pm \text{GS}_\pm \rangle$ |

Different \mathbb{Z}_2 weak SPTs

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Example: \mathbb{Z}_2 weak SPTs

Are H_+ and H_- in different \mathbb{Z}_2 weak SPT phases?

Inserting a **translation defect** is done by

$$T^L = 1 \rightarrow T^L = T \implies L \rightarrow L - 1$$

► Translation defect carries \mathbb{Z}_2 symmetry charge in $|\text{GS}_-\rangle$

Weak SPTs \leftrightarrow Translation defects dressed by SPTs

| | Even L | Even L , \mathbb{Z}_2 symmetry defect | Odd L |
|----------------------------|--------------------------|--|----------------------------|
| $U \text{GS}_\pm\rangle =$ | $+ \text{GS}_\pm\rangle$ | $+ \text{GS}_\pm\rangle$ | $\pm \text{GS}_\pm\rangle$ |
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A curious Hamiltonian

1d periodic lattice with a single **qubit** and \mathbb{Z}_4 **qudit** on each site $j \sim j + L$ [SP, Lam, Aksoy '24]

- σ^x, σ^z act on **qubits**: $(\sigma^x)^2 = (\sigma^z)^2 = 1$ and $\sigma^z \sigma^x = -\sigma^x \sigma^z$
- X, Z act on \mathbb{Z}_4 **qudits**: $X^4 = Z^4 = 1$ and $ZX = iXZ$

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$$H = \sum_j (Z_j - Z_j^\dagger) \sigma_j^z (Z_{j+1} - Z_{j+1}^\dagger) - \sigma_j^x C_{j+1} \sigma_{j+1}^x$$

- C acts as $X \rightarrow X^\dagger$ and $Z \rightarrow Z^\dagger$
- Is a sum of commuting terms and has a **unique** gapped ground state

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- X, Z act on **\mathbb{Z}_4 qudits**: $X^4 = Z^4 = 1$ and $ZX = iXZ$

$$H = \sum_j (Z_j - Z_j^\dagger) \sigma_j^z (Z_{j+1} - Z_{j+1}^\dagger) - \sigma_j^x C_{j+1} \sigma_{j+1}^x$$

- $C_2 = \frac{1}{2}(X + X^\dagger + Z + Z^\dagger)$
- $I_8 = \frac{1}{8}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4)$
- $g_1 = \frac{1}{2}(1 + Z + Z^2 + Z^3)$
- $g_2 = \frac{1}{2}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4)$
- $g_3 = \frac{1}{2}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + XZ + XZ^2 + XZ^3 + XZ^4 + ZX + ZX^2 + ZX^3 + ZX^4)$
- $g_4 = \frac{1}{8}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^2Z + X^2Z^2 + X^2Z^3 + X^2Z^4 + Z^2X + Z^2X^2 + Z^2X^3 + Z^2X^4 + XZ^2 + XZ^3 + XZ^4 + ZX^2 + ZX^3 + ZX^4)$
- $g_5 = \frac{1}{16}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z + X^3Z^2 + X^3Z^3 + X^3Z^4 + Z^3X + Z^3X^2 + Z^3X^3 + Z^3X^4 + X^2Z^2 + X^2Z^3 + X^2Z^4 + Z^2X^2 + Z^2X^3 + Z^2X^4 + XZ^3 + XZ^4 + ZX^3 + ZX^4)$
- $g_6 = \frac{1}{32}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^2 + X^3Z^3 + X^3Z^4 + Z^3X^2 + Z^3X^3 + Z^3X^4 + X^2Z^3 + X^2Z^4 + Z^2X^3 + Z^2X^4 + XZ^4 + ZX^4)$
- $g_7 = \frac{1}{64}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^3 + X^3Z^4 + Z^3X^3 + Z^3X^4 + X^2Z^4 + Z^2X^4)$
- $g_8 = \frac{1}{128}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_9 = \frac{1}{256}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4)$
- $g_{10} = \frac{1}{512}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{11} = \frac{1}{1024}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{12} = \frac{1}{2048}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{13} = \frac{1}{4096}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{14} = \frac{1}{8192}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{15} = \frac{1}{16384}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{16} = \frac{1}{32768}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{17} = \frac{1}{65536}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{18} = \frac{1}{131072}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{19} = \frac{1}{262144}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{20} = \frac{1}{524288}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{21} = \frac{1}{1048576}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{22} = \frac{1}{2097152}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{23} = \frac{1}{4194304}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{24} = \frac{1}{8388608}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{25} = \frac{1}{16777216}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{26} = \frac{1}{33554432}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{27} = \frac{1}{67108864}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{28} = \frac{1}{134217728}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{29} = \frac{1}{268435456}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{30} = \frac{1}{536870912}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{31} = \frac{1}{1073741824}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{32} = \frac{1}{2147483648}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{33} = \frac{1}{4294967296}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
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- $g_{45} = \frac{1}{17592186044416}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
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- $g_{57} = \frac{1}{72057594037927936}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{58} = \frac{1}{144115188075855872}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{59} = \frac{1}{288230376151711744}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
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- $g_{61} = \frac{1}{1152921504606846976}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{62} = \frac{1}{2305843009213693952}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{63} = \frac{1}{4611686018427387904}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{64} = \frac{1}{9223372036854775808}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{65} = \frac{1}{18446744073709551616}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{66} = \frac{1}{36893488147419103232}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{67} = \frac{1}{73786976294838206464}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{68} = \frac{1}{147573952589676412928}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{69} = \frac{1}{295147905179352825856}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{70} = \frac{1}{590295810358705651712}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{71} = \frac{1}{1180591620717411303424}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{72} = \frac{1}{2361183241434822606848}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{73} = \frac{1}{4722366482869645213696}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{74} = \frac{1}{9444732965739290427392}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{75} = \frac{1}{18889465931478580854784}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{76} = \frac{1}{37778931862957161689568}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{77} = \frac{1}{75557863725914323379136}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{78} = \frac{1}{151115727451828646758272}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{79} = \frac{1}{302231454903657293516544}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{80} = \frac{1}{604462909807314587033088}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{81} = \frac{1}{1208925819614629174066176}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{82} = \frac{1}{2417851639229258348132352}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{83} = \frac{1}{4835703278458516696264704}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{84} = \frac{1}{9671406556917033392529408}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{85} = \frac{1}{19342813113834066785058816}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{86} = \frac{1}{38685626227668133570117632}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{87} = \frac{1}{77371252455336267140235264}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{88} = \frac{1}{154742504910672534280470528}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{89} = \frac{1}{309485009821345068560941056}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{90} = \frac{1}{618970019642690137121882112}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{91} = \frac{1}{1237940039285380274243764224}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{92} = \frac{1}{2475880078570760548487528448}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{93} = \frac{1}{4951760157141521096975056896}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{94} = \frac{1}{9903520314283042193950113792}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{95} = \frac{1}{19807040628566084387900227584}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{96} = \frac{1}{39614081257132168775800455168}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{97} = \frac{1}{79228162514264337551600810336}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{98} = \frac{1}{158456325228528675103201620672}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{99} = \frac{1}{316912650457057350206403241344}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$
- $g_{100} = \frac{1}{633825300914114700412806482688}(1 + X + X^2 + X^3 + Z + Z^2 + Z^3 + Z^4 + X^3Z^4 + Z^3X^4)$

Some curious symmetries

$$H = \sum_j (Z_j - Z_j^\dagger) \sigma_j^z (Z_{j+1} - Z_{j+1}^\dagger) - \sigma_j^x C_{j+1} \sigma_{j+1}^x$$

What are the **symmetries** of H ?

Some curious symmetries

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- \mathbb{Z}_L lattice **translations** $T: j \rightarrow j + 1$
- Three \mathbb{Z}_2 symmetry operators

$$U = \prod_j X_j^2, \quad R_1 = \prod_j \sigma_j^z, \quad R_2 = \prod_j Z_j^2$$

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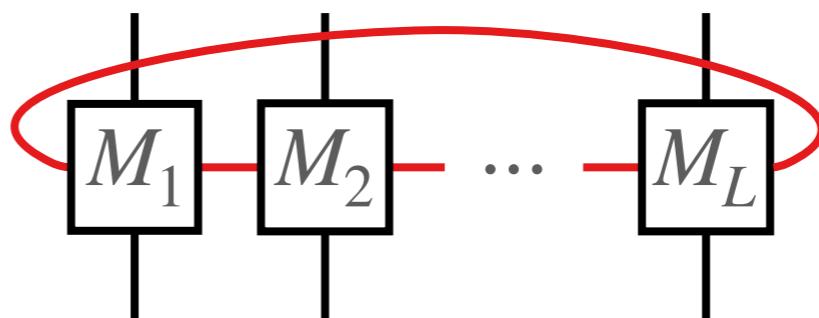
-  symmetry operator

$$R_E = \frac{1}{2} (1 + R_1) (1 + R_2) \prod_j Z_j^{\prod_{k=1}^{j-1} \sigma_k^z}$$

Some curious symmetries

R_E can be written as a $\chi = 2$ matrix product operator

$$R_E = \text{Tr} \left(\prod_{j=1}^L M_j \right) \equiv$$



► MPO tensor

$$M_j = \frac{1}{2} \begin{pmatrix} Z_j + Z_j^\dagger & i(Z_j - Z_j^\dagger) \sigma_j^z \\ -i(Z_j - Z_j^\dagger) & (Z_j + Z_j^\dagger) \sigma_j^z \end{pmatrix}$$

► 🤔 symmetry operator

$$R_E = \frac{1}{2} (1 + R_1) (1 + R_2) \prod_j Z_j^{\prod_{k=1}^{j-1} \sigma_k^z}$$

Some curious symmetries

$$H = \sum_j (Z_j - Z_j^\dagger) \sigma_j^z (Z_{j+1} - Z_{j+1}^\dagger) - \sigma_j^x C_{j+1} \sigma_{j+1}^x$$

What are the symmetries of H ?

- R_E is a **non-invertible symmetry** operator
- $R_1 |\psi\rangle = - |\psi\rangle$ or $R_2 |\psi\rangle = - |\psi\rangle \implies R_E |\psi\rangle = 0$
- R_E have zero-eigenvalues $\implies R_E$ is non-invertible
-  symmetry operator

$$R_E = \frac{1}{2} (1 + R_1) (1 + R_2) \prod_j Z_j^{\prod_{k=1}^{j-1} \sigma_k^z}$$

A curious SPT

These symmetry operators obey

$$U^2 = 1, \quad R_i^2 = 1, \quad R_E^2 = 1 + R_1 + R_2 + R_1 R_2, \quad R_E R_i = R_i R_E = R_E$$

$$UR_E = (-1)^L R_E U$$

- Form a (projective) $\mathbb{Z}_2 \times \text{Rep}(D_8)$ symmetry*

Dihedral group of order 8 $D_8 \simeq \langle r, s \mid r^2 = s^4 = 1, rsr = s^3 \rangle$

- Four 1d reps $1, P_1, P_2, P_3 = P_1 \otimes P_2$ and one 2d irrep E

$$P_i \otimes P_i = 1 \quad E \otimes E = 1 \oplus P_1 \oplus P_2 \oplus P_3 \quad E \otimes P_i = P_i \otimes E = E$$

*Confirmed $\text{Rep}(D_8)$ over other $\text{TY}(D_4)$ via gauging

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A curious SPT

These symmetry operators obey

$U^2 = 1$, H is in a $\mathbb{Z}_2 \times \text{Rep}(D_8)$ weak SPT phase

- Translation defects carry $\text{Rep}(D_8)$ symmetry charge in $|\text{GS}\rangle$
- Form a

$R_i R_E = R_E$

Ground state satisfies:

$$T|\text{GS}\rangle = +|\text{GS}\rangle$$

$$U|\text{GS}\rangle = +|\text{GS}\rangle$$

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Inserting an R_E symmetry defect

An R_E symmetry defect can be inserted using the MPO presentation of R_E

$$R_E^{(I)} = \begin{array}{c} \text{---} \\ | \\ \boxed{M_I} \end{array} \text{---} \begin{array}{c} \text{---} \\ | \\ \boxed{M_{I+1}} \end{array} \text{---} \begin{array}{c} \text{---} \\ | \\ \boxed{M_{I+2}} \end{array} \text{---} \cdots$$

- Maps states in $\mathcal{H} \cong \mathbb{C}^{8L}$ to those in $\mathcal{H}_E \cong \mathbb{C}^2 \otimes \mathcal{H}$

Defect Hamiltonian ($R_{\mathbb{F}}^{(I)}H = H_{\mathbb{F}}^{(I-1,I)}R_{\mathbb{F}}^{(I)}$)

$$H_{\mathbb{E}}^{(I-1,I)} = H + (1 - Z_{\text{defect}}) \sigma_{I-1}^x C_I \sigma_I^x$$

- Two exactly degenerate ground states

$$|GS_+\rangle = |+1\rangle \otimes |GS\rangle \quad |GS_-\rangle = |-1\rangle \otimes |\widetilde{GS}\rangle$$

Inserting an R_E symmetry defect

An R symmetry defect can be inserted using the MPO

E-twisted symmetry operators satisfy

$$T|GS_{\pm}\rangle = |GS_{\mp}\rangle \quad U|GS_{\pm}\rangle = \pm|GS_{\pm}\rangle \quad R_1|GS_{\pm}\rangle = |GS_{\pm}\rangle$$

$$R_2|GS_{\pm}\rangle = \begin{cases} +|GS_{\pm}\rangle, & L \text{ even} \\ -|GS_{\pm}\rangle, & L \text{ odd} \end{cases} \quad R_E|GS_{\pm}\rangle = \begin{cases} 2|GS_{\pm}\rangle, & L \text{ even} \\ 0, & L \text{ odd} \end{cases}$$

Defect Hamiltonian ($R_E^{(I)}H = H_E^{(I-1,I)}R_E^{(I)}$)

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A curious projective algebra

This SPT is characterized by a projective **symmetry**:

$$UR_E = -R_E U \quad (\text{odd } L)$$

Projective unitary **symmetries** $U_1 U_2 = e^{i\theta} U_2 U_1$ forbid SPTs

► Assume non-degenerate **symmetric** ground state $|\text{GS}\rangle$

$$\left. \begin{array}{l} 1. \ U_1 U_2 |\text{GS}\rangle = |\text{GS}\rangle \\ 2. \ U_1 U_2 |\text{GS}\rangle = e^{i\theta} U_2 U_1 |\text{GS}\rangle = e^{i\theta} |\text{GS}\rangle \end{array} \right\} \begin{array}{l} \text{Contradicts} \\ \text{assumption} \end{array}$$

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Projective **non-invertible symmetries** are compatible with SPTs

► **Loophole**: symmetry operator has zero-eigenvalues

$$UR_E = (-1)^L R_E U \implies R_E |\text{SPT}\rangle = 0 \text{ when } L \text{ is odd}$$

Non-invertible symmetry and SPTs

SPTs protected by internal **invertible** versus **non-invertible**

symmetry [Thorngren, Wang '19; Inamura '21; Fechisin, Tantivasadakarn, Albert '23; Antinucci, Bhardwaj, Bottini, Copetti, Gai, Huang, Pajer, Schäfer-Nameki, Tiwari, Warman, Wu '23-25; Seifnashri, Shao '24; Li, Litvinov '24; Jia '24; Inamura, Ohyama '24; Meng, Yang, Lan, Gu '24; Cao, Yamazaki, Li '25; Aksoy, Wen '25]

| Properties | Invertible | Non-invertible |
|---------------------------------------|------------|----------------|
| Stacking/Entanglers | Yes | No |
| Classification | Cobordism | Fiber functors |
| Edge/interface modes | Yes | Yes |
| Defect characterization | Yes | Yes |
| Compatible with projectivite algebras | No | Yes |

Projective $Z(G) \times \text{Rep}(G)$ symmetry

The **projective** $\mathbb{Z}_2 \times \text{Rep}(D_8)$ symmetry is a **special case** of a **projective** $Z(G) \times \text{Rep}(G)$ symmetry

- $Z(G)$ is the center of a finite group G
- $\text{Rep}(G)$ is the fusion category of representations of G

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- $Z(G)$ is the center of a finite group G
- $\text{Rep}(G)$ is the fusion category of representations of G

Onsite $Z(G)$ **symmetry** operator $U_z = \prod_j U_j^{(z)}$, with $z \in Z(G)$:

$$U_{z_1} U_{z_2} = U_{z_1 z_2}$$

$\text{Rep}(G)$ **symmetry** operator R_Γ , with Γ an irrep of G :

$$R_{\Gamma_a} \times R_{\Gamma_b} = \sum_c N_{ab}^c R_{\Gamma_c}$$

- **Non-invertible symmetry** when G is non-Abelian

Projective $Z(G) \times \text{Rep}(G)$ symmetry

The **projectivity** arises through the local relation

$$R_\Gamma U_j^{(z)} = e^{i\phi_\Gamma(z)} U_j^{(z)} R_\Gamma \text{ with } e^{i\phi_\Gamma(z)} = \text{Tr}[\Gamma(z)] / d_\Gamma$$

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e.g., $e^{i\phi_\Gamma(z)}$ when $G = \mathbb{Z}_2$ ($Z(\mathbb{Z}_2) = \mathbb{Z}_2$)

| Γ | 1 | sign |
|----------|------|------|
| z | $+1$ | $+1$ |
| | -1 | $+1$ |

- The symmetries of XY model we saw in morning talk

$$R_{\text{sign}} = \prod_{j=1}^L Z_j \quad U_{-1} = \prod_{j=1}^L X_j$$

Projective $Z(G) \times \text{Rep}(G)$ symmetry

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e.g., $e^{i\phi_\Gamma(z)}$ when $\mathbf{G} = \mathbf{D}_8$ ($Z(D_8) = \mathbb{Z}_2$)

| Γ | 1 | 1_1 | 1_2 | 1_3 | E |
|----------|----|-------|-------|-------|----|
| z | +1 | +1 | +1 | +1 | +1 |
| | -1 | +1 | +1 | +1 | +1 |

Projective $Z(G) \times \text{Rep}(G)$ symmetry

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| Γ | 1 | 1_1 | 1_2 | 1_3 | E |
|----------|----|-------|-------|-------|-----|
| z | +1 | +1 | +1 | +1 | +1 |
| | -1 | +1 | +1 | +1 | -1 |

Explicit expressions of U_z and R_Γ for the Hilbert space $\bigotimes_j \mathbb{C}^{|G|}$

$$U_z = \sum_{\{g_j\}} |zg_1, \dots, zg_L\rangle \langle g_1, \dots, g_L| \quad R_\Gamma = \sum_{\{g_j\}} \text{Tr}[\Gamma(g_1 \cdots g_L)] |g_1, \dots, g_L\rangle \langle g_1, \dots, g_L|$$

Constraints from projectivity

The local projective algebra implies $R_\Gamma U_z = (e^{i\phi_\Gamma(z)})^L U_z R_\Gamma$

- When $e^{i\phi_\Gamma(z)}$ is non-trivial for a unitary R_Γ , this is a manifestation of a **Lieb-Schultz-Mattis (LSM) anomaly**
- The LSM anomaly forbids **SPT** phases

[Lieb, Schultz, Mattis '61; Oshikawa '99; Hastings '03; ...; Chen, Gu, Wen '10; Else, Thorngren '19; Yao, Oshikawa '20; Ogata, Tasaki '21; Cheng, Seiberg '22; Seifnashri '23; Kapustin, Sopenko '24]

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When $e^{i\phi_\Gamma(z)}$ is non-trivial for only non-invertible R_Γ , there is the $R_\Gamma | \text{SPT} \rangle = 0$ loophole \implies Can have an **SPT**,

- Does this **projective algebra** then have any consequences?

Yes! There is an **SPT-LSM theorem**

SPT-LSM theorems

An **SPT-LSM** theorem is an obstruction to a trivial **SPT***

[Lu '17; Yang, Jiang, Vishwanath, Ran '17; Lu, Ran, Oshikawa '17; Else, Thorngren '19; Jiang, Cheng, Qi, Lu '19]

- Any **SPT state** must have non-zero entanglement

Symmetry-enforced entanglement

*Trivial SPT = symmetric product state, which is a non-canonical choice

SPT-LSM theorems

An **SPT-LSM** theorem is an obstruction to a trivial **SPT***

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- Any **SPT state** must have non-zero entanglement

Symmetry-enforced entanglement

Why does the **projective algebra**

$$R_\Gamma U_z = (e^{i\phi_\Gamma(z)})^L U_z R_\Gamma$$

gives rise to an **SPT-LSM theorem**?

- Local projective algebra forbids a trivial **SPT**
- Any $|SPT\rangle$ must satisfy $R_\Gamma |SPT\rangle = 0$ when $(e^{i\phi_\Gamma(z)})^L \neq 1$

*Trivial SPT = symmetric product state, which is a non-canonical choice

Simple SPT-LSM example

Consider a 1 + 1D system with two \mathbb{Z}_4 qudits on each site $j \sim j + L$ with L even and $\mathbb{Z}_4 \times \mathbb{Z}_4$ symmetry operators

$$U = \prod_j X_j \tilde{X}_j \quad V = \prod_j (Z_j \tilde{Z}_j)^{2j+1}$$

Local projective algebra $U_j V_j = - V_j U_j$, but no LSM anomaly

[Jiang, Cheng, Qi, Lu '19]

- Defect perspective: Inserting a U symmetry defect causes

$$T_{\text{tw}} V = \left(- \prod_j Z_j^2 \tilde{Z}_j^2 \right) V T_{\text{tw}}$$

*Non-abelian group,
not a projective rep!*

Furthermore, there is no trivial $|\text{SPT}\rangle = \bigotimes_j |\psi_j\rangle$

- Easily proven by contradiction using $U_j V_j = - V_j U_j$

Simple SPT-LSM example

Consider a $1 + 1$ D system with two \mathbb{Z}_4 qudits on each site $j \sim j + L$ with L even and $\mathbb{Z}_4 \times \mathbb{Z}_4$ symmetry operators

$$U = \prod_j X_j \tilde{X}_j \quad V = \prod_j (Z_j \tilde{Z}_j)^{2j+1}$$

Local projective algebra $U_j V_j = -V_j U_j$, but no LSM anomaly

[Liu, Cheng, Qi, Lu '19]

► Defects cause

Key point: obstruction to **product state**

SPT while keeping U and V onsite

► Common for **modulated SPTs** [SP, work in progress]

Furthermore, there is no trivial SPT $1 = \bigotimes_j |\Psi_j\rangle$

► Easily proven by contradiction using $U_j V_j = -V_j U_j$

SPT-LSM theorem proof

To prove our **SPT-LSM theorem**, we

1. Use that the $Z(G)$ symmetry is on-site:

$$U_z = \prod_j U_j^{(z)} \text{ which satisfies } R_\Gamma U_j^{(z)} = e^{i\phi_\Gamma(z)} U_j^{(z)} R_\Gamma$$

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2. Use that any translation-inv product state $|\text{GS}\rangle$ satisfies

$$R_\Gamma |\text{GS}\rangle \neq 0 \text{ for some } L = L^* \left(e^{i\phi_\Gamma(z)L^*} = 1 \right)$$

$= d_\Gamma$ for $L = |G|\mathbb{Z}$

► For $\mathcal{H}_j = \mathbb{C}^{|G|}$, $R_\Gamma \bigotimes_{j=1}^L \sum_{g \in G} c_g |g\rangle = \chi_\Gamma(\tilde{g}^L) c_{\tilde{g}}^L |\tilde{g} \cdots \tilde{g}\rangle + \cdots$

► Generally true if there is an IR **TQFT** description since

$$R_\Gamma |\text{GS}_{\text{TQFT}}\rangle = d_\Gamma |\text{GS}_{\text{TQFT}}\rangle$$

SPT-LSM theorem proof

If there is an SPT state $|\text{GS}\rangle$ that is a **product state**:

► $U_z |\text{GS}\rangle = |\text{GS}\rangle \implies U_j^{(z)} |\text{GS}\rangle = |\text{GS}\rangle$

Using that $R_\Gamma |\text{GS}\rangle = \lambda_\Gamma |\text{GS}\rangle \neq 0$ at $L = L^*$:

1. $R_\Gamma U_j^{(z)} |\text{GS}\rangle = R_\Gamma |\text{GS}\rangle = \lambda_\Gamma |\text{GS}\rangle \xleftarrow{\hspace{10em}} \text{Contradiction}$

2. $R_\Gamma U_j^{(z)} |\text{GS}\rangle = e^{i\phi_\Gamma(z)} U_j^{(z)} R_\Gamma |\text{GS}\rangle = \lambda_\Gamma e^{i\phi_\Gamma(z)} |\text{GS}\rangle \xleftarrow{\hspace{10em}}$

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⇒ Cannot be an SPT state that is a **product state** at $L = L^*$

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⇒ Cannot be an **SPT state** that is a **product state** at $L = L^*$

⇒ By locality, there cannot be an **SPT state** that is a **product state** for any L

SPT-LSM theorem proof

If there is an SPT state $|\text{GS}\rangle$ that is a product state:

$$\blacktriangleright U_z |\text{GS}\rangle = |\text{GS}\rangle \implies U_j^{(z)} |\text{GS}\rangle = |\text{GS}\rangle$$

Using $U_j^{(z)} = U_z$ (locality)

Therefore, the **projective non-invertible symmetry**

1. $U_z |\text{GS}\rangle = |\text{GS}\rangle$ prevents a product state SPT

2. $U_z^{(z)} |\text{GS}\rangle = |\text{GS}\rangle$ \blacktriangleright All SPTs must have **non-zero entanglement**

\implies Cannot be an SPT state that is a product state at $L = L^*$

\implies By locality, there cannot be an SPT state that is a product state for any L

Non-invertible weak SPT

What is the characterization of these SPTs?

- They must satisfy $\mathcal{R}_\Gamma |\text{GS}\rangle = 0$ for nontrivial $(e^{i\phi_\Gamma(z)})^L$

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Two possibilities:

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The first is incompatible with an IR TQFT

Non-invertible weak SPT

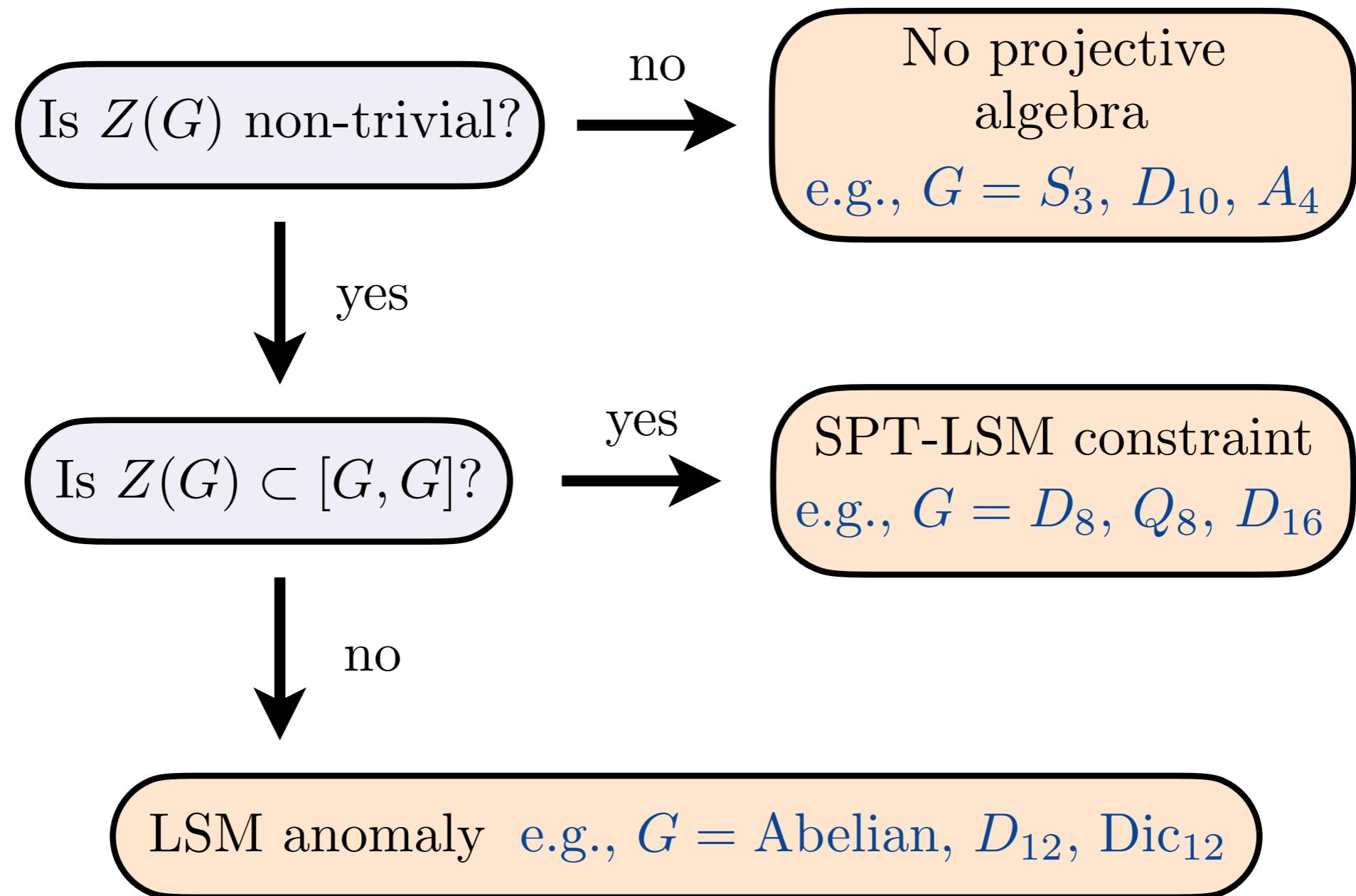
What is the characterization of these SPTs?

- Theorem 1: $\text{SPT} \rightarrow \text{SPT}$ (non-invertible) $\Leftrightarrow \text{SPT} \rightarrow \text{SPT}$ (invertible) $\Leftrightarrow L = L^*$
- At $L = L^*$, SPTs satisfy $R_\Gamma |\text{GS}\rangle = \lambda_\Gamma |\text{GS}\rangle \neq 0$
- At $L = L^* + 1$, SPTs satisfy $R_\Gamma |\text{GS}\rangle = 0$
- All SPT states have translation defects dressed by non-trivial $\text{Rep}(G)$ symmetry charge
- \exists a trivial SPT \implies SPT-LSM theorem

The first is incompatible with an IR TQFT

(SPT)-LSM theorems

Whether there is an **(SPT)-LSM theorem** depends on G :



Outlook

We found a new class of entangled weak SPTs characterized by a projective $Z(G) \times \text{Rep}(G)$ **non-invertible symmetry**

1. An exactly solvable model in a **weak SPT phase** characterized by a projective $\mathbb{Z}_2 \times \text{Rep}(D_8)$ **symmetry**
2. General discussion on projective $Z(G) \times \text{Rep}(G)$ **weak SPTs**
 \implies an **SPT-LSM theorem**

For the newcomer: New **quantum phases** and models can be discovered using **generalized symmetries** as a guide!

For the initiated: **Beyond-relativistic-QFT-symmetries** are interesting!

Back-up slides

The surprising lack of an 't Hooft anomaly

Inserting U or R_E symmetry defects leads to the **projective algebras**

| U symmetry defect | R_E symmetry defect |
|---------------------|-----------------------|
| $R_E T = -TR_E$ | $TU = -UT$ |

For invertible symmetries, such **projective algebras** imply an 't Hooft anomaly (e.g., the type III anomaly $(-1)^{\int_{M_3} a \cup b \cup c}$)

[Matsui '08; Yao, Oshikawa '20; Seifnashri '23; Kapustin, Sopenko '24]

- This is not true for **non-invertible symmetries!**

The surprising lack of an 't Hooft anomaly

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[; Kapustin, Sopenko '24]

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For invertible symmetries, such as the type A symmetries; Kapustin, Sopena, Wil�czek; [arXiv:1810.08036](#)

Fails because of $R_E = 0$ loophole

Fails because the degeneracy is encoded in the defect's **quantum dimension**

Projective $\mathbb{Z}_2 \times \text{Rep}(D_8)$ bond algebra

$$\mathfrak{B} [\text{Rep}(D_8) \times \mathbb{Z}_2] = \left\langle \sigma_j^z, \ Z_j^2, \ Z_j Z_{j+1}, \ \sigma_j^x C_{j+1} \sigma_{j+1}^x, \ X_j^{\sigma_j^z} X_{j+1}^\dagger \right\rangle$$

Example: $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPTs

1d closed chain in space with **two qubits** on each site $j \sim j + L$ acted on by **Pauli operators** X_j, Z_j and \tilde{X}_j, \tilde{Z}_j .

$$H_p = - \sum_{j=1}^L (X_j + \tilde{X}_j)$$

$$H_c = - \sum_{j=1}^L (\tilde{Z}_{j-1} X_j \tilde{Z}_j + Z_j \tilde{X}_j Z_{j+1})$$

$$|\text{GS}_p\rangle = |+++ \cdots +\rangle$$

$$|\text{GS}_c\rangle = \tilde{Z}_{j-1} X_j \tilde{Z}_j |\text{GS}_c\rangle = Z_j \tilde{X}_j Z_{j+1} |\text{GS}_c\rangle$$

- Both models have a **unique symmetric gapped ground state**
- There is a $\mathbb{Z}_2 \times \tilde{\mathbb{Z}}_2$ **symmetry** $U = \prod_j X_j$ and $\tilde{U} = \prod_j \tilde{X}_j$ with $U|\text{GS}_.\rangle = \tilde{U}|\text{GS}_.\rangle = |\text{GS}_.\rangle$

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- Both models have a unique symmetric gapped ground state
- H_p and H_c are both in a $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPT phase
- There is a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry $U = \prod_j X_j$ and $\tilde{U} = \prod_j \tilde{X}_j$ with $U|\text{GS}_p\rangle = \tilde{U}|\text{GS}_c\rangle = |\text{GS}_p\rangle$

Distinguishing $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPTs

Are H_p and H_c in different $\mathbb{Z}_2 \times \tilde{\mathbb{Z}}_2$ SPT phases?

We can check by inserting a $\textcolor{teal}{U}$ symmetry defect at $\langle L, 1 \rangle$

- Gives rise to $\textcolor{brown}{U}$ -twisted boundary conditions: $Z_{j+L} = -Z_j$
- 1. H_p is unaffected, so its ground state still satisfies

$$U|\text{GS}_{p;U}\rangle = +|\text{GS}_{p;U}\rangle \quad \tilde{U}|\text{GS}_{p;U}\rangle = +|\text{GS}_{p;U}\rangle$$

- 2. H_c becomes $\textcolor{teal}{H}_c + 2Z_L \tilde{X}_L Z_1$, and its ground state satisfies

$$U|\text{GS}_{c;U}\rangle = +|\text{GS}_{c;U}\rangle \quad \tilde{U}|\text{GS}_{c;U}\rangle = -|\text{GS}_{c;U}\rangle$$

Distinguishing $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPTs

Different **responses** imply that H_p and H_c are in different $\mathbb{Z}_2 \times \tilde{\mathbb{Z}}_2$ SPT phases

[Chen, Lu, Vishwanath 2013; Gaiotto, Johnson-Freyd 2017; Wang, Ning, Cheng 2021]

Low-energy EFTs of H_p and H_c

$$Z_p[A, \tilde{A}] = 1 \quad Z_c[A, \tilde{A}] = (-1)^{\int A \cup \tilde{A}}$$

1. H_p is **unaffected**, so its ground state still satisfies

$$U|\text{GS}_{p;U}\rangle = +|\text{GS}_{p;U}\rangle \quad \tilde{U}|\text{GS}_{p;U}\rangle = +|\text{GS}_{p;U}\rangle$$

2. H_c becomes $H_c + 2Z_L \tilde{X}_L Z_1$, and its ground state satisfies

$$U|\text{GS}_{c;U}\rangle = +|\text{GS}_{c;U}\rangle \quad \tilde{U}|\text{GS}_{c;U}\rangle = -|\text{GS}_{c;U}\rangle$$

LSM anomaly in the XY model

Many-qubit model on a periodic chain with Hamiltonian

$$H = \sum_{j=1}^L J \sigma_j^x \sigma_{j+1}^x + K \sigma_j^y \sigma_{j+1}^y$$

- There is an **LSM anomaly** involving the $\mathbb{Z}_2^x \times \mathbb{Z}_2^y \times \mathbb{Z}_L$ symmetry [Chen, Gu, Wen 2010; Ogata, Tasaki 2021]

$$U_x = \prod_j \sigma_j^x, \quad U_y = \prod_j \sigma_j^y, \quad \text{and lattice translations } T$$

- Manifests through the **projective algebras** [Cheng, Seiberg 2023]

| <i>Translation defects</i> | \mathbb{Z}_2^x defect | \mathbb{Z}_2^y defect |
|----------------------------|-------------------------|-------------------------|
| $U_x U_y = (-1)^L U_y U_x$ | $U_y T = - T U_y$ | $T U_x = - U_x T$ |

GROUP BASED QUDITS

A **G -qudit** is a $|G|$ -level quantum mechanical system whose states are $|g\rangle$ with $g \in G$

- G is a **finite group**, e.g. $\mathbb{Z}_2, S_3, D_8, \text{SmallGroup}(32,49)$

Group based **Pauli operators** [Brell 2014]

- X operators labeled by group elements

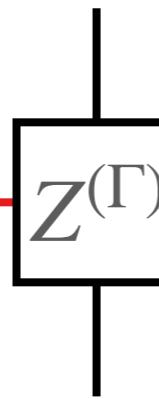
$$\vec{X}^{(g)} = \sum_h |gh\rangle\langle h|$$

$$\overleftarrow{X}^{(g)} = \sum_h |h\bar{g}\rangle\langle h|$$

$$\bar{g} \equiv g^{-1}$$

- Z operators are MPOs labeled by **irreps** $\Gamma: G \rightarrow \text{GL}(d_\Gamma, \mathbb{C})$

$$[Z^{(\Gamma)}]_{\alpha\beta} = \sum_h [\Gamma(h)]_{\alpha\beta} |h\rangle\langle h| \equiv \alpha \text{---} Z^{(\Gamma)} \text{---} \beta \quad (\alpha, \beta = 1, 2, \dots, d_\Gamma)$$



GROUP BASED QUDITS

Example: $G = \mathbb{Z}_2$ where $g \in \{1, -1\}$ and $\Gamma \in \{1, 1'\}$

$$\vec{X}^{(1)} = \overleftarrow{X}^{(1)} = [Z^{(1)}]_{11} = 1$$

$$\vec{X}^{(-1)} = \overleftarrow{X}^{(-1)} = \sigma^x \quad [Z^{(1')}]_{11} = \sigma^z$$

Group based Pauli operators satisfy

1. $\vec{X}^{(g)} \vec{X}^{(h)} = \vec{X}^{(gh)}$, $\overleftarrow{X}^{(g)} \overleftarrow{X}^{(h)} = \overleftarrow{X}^{(gh)}$, and $\vec{X}^{(g)} \overleftarrow{X}^{(h)} = \overleftarrow{X}^{(h)} \vec{X}^{(g)}$
2. $\vec{X}^{(g)} \vec{X}^{(h)} = \vec{X}^{(h)} \vec{X}^{(g)}$ iff g and h commute
3. $\vec{X}^{(g)} [Z^{(\Gamma)}]_{\alpha\beta} = [\Gamma(\bar{g})]_{\alpha\gamma} [Z^{(\Gamma)}]_{\gamma\beta} \vec{X}^{(g)}$
4. **Unitarity**: $\vec{X}^{(g)\dagger} = \vec{X}^{(\bar{g})}$, $\overleftarrow{X}^{(g)\dagger} = \overleftarrow{X}^{(\bar{g})}$, $[Z^{(\Gamma)\dagger} Z^{(\Gamma)}]_{\alpha\beta} = \delta_{\alpha\beta}$

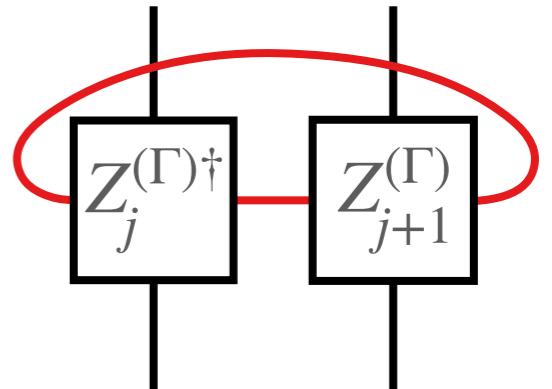
GROUP BASED XY MODEL

Group based **Pauli operators** are useful for constructing quantum lattice models [Brell 2014; Albert *et. al.* 2021; Fechisin, Tantivasadakarn, Albert 2023]

Group based **XY model**: Consider a **periodic 1d lattice** of L sites. On each site j resides a **G -qudit** and its Hamiltonian

$$H_{XY} = \sum_{j=1}^L \left(\sum_{\Gamma} J_{\Gamma} \text{Tr} \left(Z_j^{(\Gamma)\dagger} Z_{j+1}^{(\Gamma)} \right) + \sum_g K_g \overleftarrow{X}_j^{(g)} \overrightarrow{X}_{j+1}^{(g)} \right) + \text{hc}$$

$$\text{Tr} \left(Z_j^{(\Gamma)\dagger} Z_{j+1}^{(\Gamma)} \right) = \sum_{\{g\}} \chi_{\Gamma}(\bar{g}_j g_{j+1}) | \{g\} \rangle \langle \{g\} | \equiv$$



- For $G = \mathbb{Z}_2$, this is the ordinary quantum **XY model**

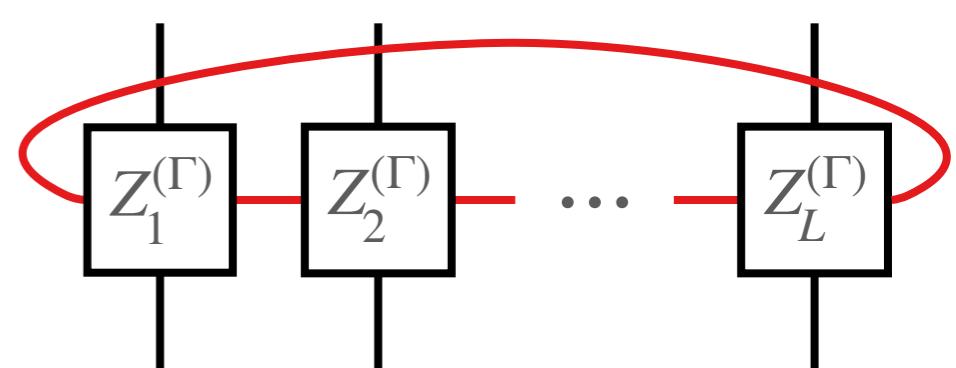
SYMMETRY OPERATORS

$$H_{XY} = \sum_{j=1}^L \left(\sum_{\Gamma} J_{\Gamma} \text{Tr} \left(Z_j^{(\Gamma)\dagger} Z_{j+1}^{(\Gamma)} \right) + \sum_g K_g \overleftarrow{X}_j^{(g)} \overrightarrow{X}_{j+1}^{(g)} \right) + \text{hc}$$

\mathbb{Z}_L lattice translations: $T \mathcal{O}_j T^\dagger = \mathcal{O}_{j+1}$

Various internal symmetries:

► $Z(G)$ symmetry $U_z = \prod_j \overrightarrow{X}_j^{(z)}$ with $z \in Z(G)$

► $\text{Rep}(G)$ symmetry $R_{\Gamma} = \text{Tr} \left(\prod_{j=1}^L Z_j^{(\Gamma)} \right) \equiv$ 

$$R_{\Gamma_a} \times R_{\Gamma_b} = R_{\Gamma_a \otimes \Gamma_b} = R_{\bigoplus_c N_{ab}^c \Gamma_c} = \sum_c N_{ab}^c R_{\Gamma_c}$$

PROJECTIVE ALGEBRA FROM DEFECTS

$$U_z = \prod_j \vec{X}_j^{(z)}$$

$$T_{\text{tw}}^{(z)} = \vec{X}_I^{(z)} T$$

$$R_\Gamma = \text{Tr} \left(\prod_{j=1}^L Z_j^{(\Gamma)} \right)$$

$$T_{\text{tw}}^{(\Gamma)} = \hat{Z}_I^{(\Gamma)} (T \otimes \mathbf{1})$$

Letting $e^{i\phi_\Gamma(z)} \equiv \chi_\Gamma(z)/d_\Gamma$

| <i>Translation defects</i> | $z \in Z(G)$ defect | $\Gamma \in \text{Rep}(G)$ defect |
|---|---|---|
| $R_\Gamma U_z = (e^{i\phi_\Gamma(z)})^L U_z R_\Gamma$ | $R_\Gamma T_{\text{tw}}^{(z)} = e^{i\phi_\Gamma(z)} T_{\text{tw}}^{(z)} R_\Gamma$ | $T_{\text{tw}}^{(\Gamma)} U_z = e^{i\phi_\Gamma(z)} U_z T_{\text{tw}}^{(\Gamma)}$ |

- Generalizes the $G = \mathbb{Z}_2$ **projective algebra** of the ordinary quantum XY model

GAUGING WEB

