An SPT-LSM theorem for weak SPTs with non-invertible symmetry

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Quantum phases and symmetry

A fundamental problem in QFT/CMT/HEP is to understand quantum phases

1. How do we diagnose different quantum phases?
2. What are the allowed possible quantum phases?

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Sometimes, phases are characterized by a symmetry

- Superfluids by U(1) boson number conservation
- Topological insulators by $\mathrm{U}(1)_f$ and $\mathbb{Z}_2^{\mathrm{T}}$ symmetries

For such phases, symmetries provide answers to questions (1) and (2).

Quantum phases and symmetry

Quantum phases $\Longleftrightarrow$ Generalized symmetries

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$$\text{Quantum phases} \Longleftrightarrow \text{Generalized symmetries}$$

Build-a-phase recipe

(1) Choose your generalized symmetries adjectives

$$\mathrm{a}_1\text{–}\mathrm{a}_2\text{–}\mathrm{a}_3\text{–}\cdots\ \text{Symmetry}$$

- e.g., *n*-form, (non-)invertible, subsystem, dipole, …

(2) Specify SSB and SPT pattern

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Phases we have yet to name!

Quantum phases and symmetry

Which quantum phases are characterized by generalized symmetries?

Why care?

1. Provides a novel and unifying perspective of quantum phases
2. Guides us towards new quantum phases and models
3. Further develops a classification of quantum phases based on symmetries (a "generalized Landau paradigm")

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TL;DR for this talk

This talk: $1+1$D SPT phases characterized by translation and non-invertible symmetries

- Find a new class of entangled weak SPTs characterized by projective non-invertible symmetries on the lattice

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- Find a new class of entangled weak SPTs characterized by projective non-invertible symmetries on the lattice

Outline

1. Review SPTs from a symmetry defect perspective
2. Simple example of an entangled weak SPT characterized by a projective non-invertible symmetry
3. General discussion on projective $Z(G)\times\mathsf{Rep}(G)$ symmetry and (SPT-)LSM theorems

What are SPTs

An SPT phase is a gapped quantum phase protected by a symmetry with a unique ground state on all closed spatial manifolds [Chen, Gu, Liu, Wen 2011; Else, Nayak 2014; ⋯]

- Interesting physics can arise on boundaries and interfaces between SPTs (e.g., topological order, gapless excitations)

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- Background gauge fields and symmetry defects

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<u>Ordinary insulator</u>

$$S[A] = \frac{1}{2}\int F \wedge \star F$$

<u>Topological insulator</u>

[Qi, Hughes, Zhang 2008; ⋯]

$$S[A] = \frac{1}{2}\int F \wedge \star F + \frac{\pi}{4\pi} F \wedge F$$

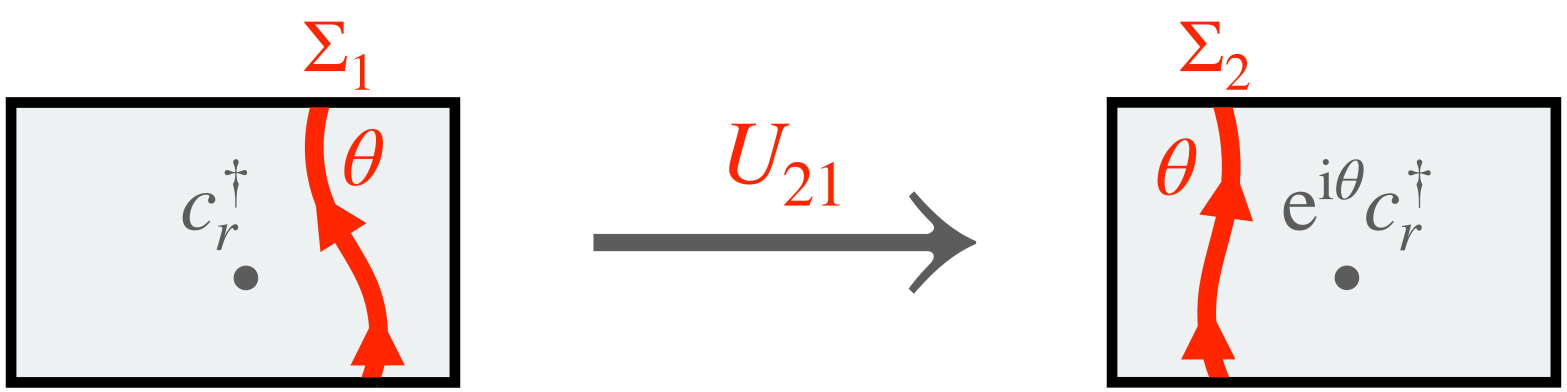
Symmetry defects

Symmetry defects are localized modifications to the Hamiltonian $H^{(\Sigma)}_{\text{defect}} = H + \delta H(\Sigma)$ and other operators

- Moved using unitary operators (are topological defects)

$$H^{(\Sigma_2)}_{\text{defect}} = U_{21} H^{(\Sigma_1)}_{\text{defect}} U^{\dagger}_{21}$$

- Implement the symmetry transformation across space



- Twisted boundary conditions $(T_{\perp})^L$ = Symmetry operator

Example: $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPTs

1d closed chain in space with two qubits on each site $j \sim j + L$ acted on by Pauli operators X_j, Z_j and $\tilde{X}_j$, $\tilde{Z}_j$.

$$H_p = -\sum_{j=1}^{L} (X_j + \tilde{X}_j)$$

$$|\mathrm{GS}_p\rangle = |+ + \cdots +\rangle$$

$$H_c = -\sum_{j=1}^{L} \left(\tilde{Z}_{j-1} X_j \tilde{Z}_j + Z_j \tilde{X}_j Z_{j+1}\right)$$

$$|\mathrm{GS}_c\rangle = \tilde{Z}_{j-1} X_j \tilde{Z}_j |\mathrm{GS}_c\rangle = Z_j \tilde{X}_j Z_{j+1} |\mathrm{GS}_c\rangle$$

- Both models have a unique symmetric gapped ground state
- There is a $\mathbb{Z}_2 \times \tilde{\mathbb{Z}}_2$ symmetry $U = \prod_j X_j$ and $\tilde{U} = \prod_j \tilde{X}_j$ with $U|\mathrm{GS}_\bullet\rangle = \tilde{U}|\mathrm{GS}_\bullet\rangle = |\mathrm{GS}_\bullet\rangle$

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H_p and H_c are both in a $\mathbb{Z}_2 \times \tilde{\mathbb{Z}}_2$ SPT phase

Distinguishing $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPTs

Are H_p and H_c in different $\mathbb{Z}_2 \times \tilde{\mathbb{Z}}_2$ SPT phases?

We can check by inserting a U symmetry defect at $\langle L,1 \rangle$

➤ Gives rise to U-twisted boundary conditions: $Z_{j+L} = -Z_j$

1. H_p is unaffected, so its ground state still satisfies

$$U \,|\, \mathrm{GS}_{p;U} \rangle = + \,|\, \mathrm{GS}_{p;U} \rangle \qquad\qquad \tilde{U} \,|\, \mathrm{GS}_{p;U} \rangle = + \,|\, \mathrm{GS}_{p;U} \rangle$$

2. H_c becomes $H_c + 2\, Z_L \, \tilde{X}_L \, Z_1$, and its ground state satisfies

$$U \,|\, \mathrm{GS}_{c;U} \rangle = + \,|\, \mathrm{GS}_{c;U} \rangle \qquad\qquad \tilde{U} \,|\, \mathrm{GS}_{c;U} \rangle = - \,|\, \mathrm{GS}_{c;U} \rangle$$

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We

➤ C

Different responses imply that H_p and H_c are in different $\mathbb{Z}_2 \times \tilde{\mathbb{Z}}_2$ SPT phases

[Chen, Lu, Vishwanath 2013; Gaiotto, Johnson-Freyd 2017; Wang, Ning, Cheng 2021]

Low-energy EFTs of H_p and H_c

$$Z_p[A,\tilde{A}] = 1 \qquad Z_c[A,\tilde{A}] = (-1)^{\int A \cup \tilde{A}}$$

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Example: $\mathbb{Z}_2$ weak SPTs

1d periodic lattice with a qubit on each site $j \sim j + L$

$$H_+ = -\sum_j X_j \quad \text{vs.} \quad H_- = +\sum_j X_j$$

- Both have a unique gapped ground state $|\mathrm{GS}_\pm\rangle = \otimes_j |\pm\rangle$
- Symmetries: $\mathbb{Z}_2 \times \mathbb{Z}_L$ with $U = \prod_j X_j$ and $T: j \to j+1$

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SPTs characterized by translations are called weak SPTs

$\boxed{H_+ \text{ and } H_- \text{ are both in } \mathbb{Z}_2 \text{ weak SPT phases}}$

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Let's insert a $U = \prod_j X_j$ symmetry defect at $\langle L, 1 \rangle$

- Neither H_+ or H_- are modified by $Z_{j+L} = -Z_j$
- Translation operator becomes $T = X_1 T_{\text{defect-free}}$ $(T^L = U)$

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	Even L	Even L, $\mathbb{Z}_2$ symmetry defect
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Different $\mathbb{Z}_2$ weak SPTs

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Translation defect carries $\mathbb{Z}_2$ symmetry charge in $|\mathrm{GS}_-\rangle$

➤ Inserting a translation defect is done by

$$T^L = 1 \to T^L = T \implies L \to L - 1$$

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A curious Hamiltonian

1d periodic lattice with a single qubit and $\mathbb{Z}_4$ qudit on each site $j \sim j + L$ [**SP**, Lam, Aksoy arXiv:2409.18113]

- σ^x, σ^z act on qubits: $(\sigma^x)^2 = (\sigma^z)^2 = 1$ and $\sigma^z\sigma^x = -\sigma^x\sigma^z$
- X, Z act on $\mathbb{Z}_4$ qudits: $X^4 = Z^4 = 1$ and $ZX = \mathrm{i}\, XZ$

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$$H = -\sum_j \sigma_j^x\, C_{j+1}\, \sigma_{j+1}^x + \frac{1}{4} \sum_j (Z_j - Z_j^\dagger)\, \sigma_j^z\, (Z_{j+1} - Z_{j+1}^\dagger)$$

- C acts as $X \to X^\dagger$ and $Z \to Z^\dagger$
- Is a sum of commuting terms and has a unique gapped ground state

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gr

$$|\mathrm{GS}\rangle = \sum_{\substack{\{\varphi_j = 0,1\} \\ \{\alpha_j = 0,2\}}} \mathrm{i}^{\sum_j \alpha_j (\varphi_j - \varphi_{j-1})} \bigotimes_j |\sigma_j^x = (-1)^{\varphi_j},\, Z_j = \mathrm{i}^{\alpha_j + 1}\rangle$$

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$$H = -\sum_j \sigma_j^x \, C_{j+1} \, \sigma_{j+1}^x + \frac{1}{4} \sum_j (Z_j - Z_j^\dagger) \, \sigma_j^z \, (Z_{j+1} - Z_{j+1}^\dagger)$$

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What are the symmetries of H?

- $\mathbb{Z}_L$ lattice translations T: $j \to j + 1$
- Three $\mathbb{Z}_2$ symmetry operators

$$U = \prod_j X_j^2, \qquad R_1 = \prod_j \sigma_j^z, \qquad R_2 = \prod_j Z_j^2$$

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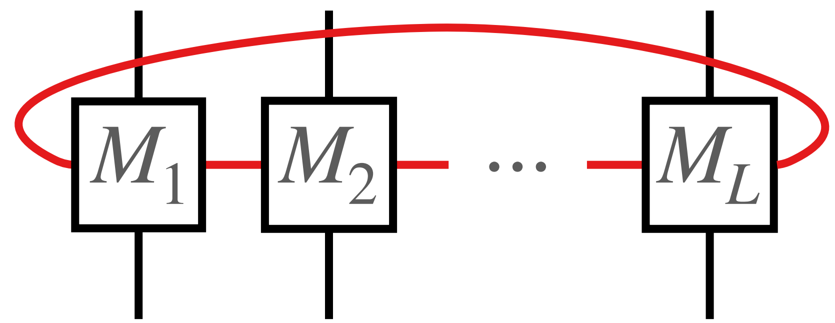
- 🤷‍♂️ symmetry operator

$$R_{\mathsf{E}} = \frac{1}{2} \left(1 + R_1\right) \left(1 + R_2\right) \prod_j Z_j^{\prod_{k=1}^{j-1} \sigma_k^z}$$

Some curious symmetries

R_{E} can be written as a $\chi = 2$ matrix product operator

$$R_{\mathsf{E}} = \mathrm{Tr}\Big(\prod_{j=1}^{L} M_j\Big) \equiv$$



- MPO tensor

$$M_j = \frac{1}{2}\begin{pmatrix} Z_j + Z_j^{\dagger} & \mathrm{i}\,(Z_j - Z_j^{\dagger})\,\sigma_j^z \\ -\mathrm{i}\,(Z_j - Z_j^{\dagger}) & (Z_j + Z_j^{\dagger})\,\sigma_j^z \end{pmatrix}$$

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R_E is a non-invertible symmetry operator

- $R_1\,|\psi\rangle = -\,|\psi\rangle$ or $R_2\,|\psi\rangle = -\,|\psi\rangle \implies R_\mathsf{E}\,|\psi\rangle = 0$
- R_E have zero-eigenvalues $\implies R_\mathsf{E}$ is non-invertible

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A curious SPT

These symmetry operators obey

$$U^2 = 1, \quad R_i^2 = 1, \quad R_{\mathsf{E}}^2 = 1 + R_1 + R_2 + R_1 R_2, \quad R_{\mathsf{E}} R_i = R_i R_{\mathsf{E}} = R_{\mathsf{E}}$$

$$U\, R_{\mathsf{E}} = (-1)^L R_{\mathsf{E}}\, U$$

- Form a (projective) $\mathbb{Z}_2 \times \mathsf{Rep}(D_8)$ symmetry

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H is in a $\mathbb{Z}_2 \times \mathsf{Rep}(D_8)$ weak SPT phase

- Translation defects carry $\mathsf{Rep}(D_8)$ symmetry charge in $|\mathrm{GS}\rangle$
- *Spoiler*: $R_{\mathsf{E}}|\mathrm{GS}\rangle = 0$ for odd $L \implies$ SPT-LSM theorem

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A curious projective algebra

This SPT is characterized by a projective symmetry:

$$U\, R_{\mathsf{E}} = -R_{\mathsf{E}}\, U \qquad \text{(odd } L\text{)}$$

Projective unitary symmetries $U_1 U_2 = \mathrm{e}^{\mathrm{i}\theta} U_2 U_1$ forbid SPTs

- Assume non-degenerate symmetric ground state:

$$\left.\begin{array}{l} 1.\ \ U_1 U_2 \,|\psi\rangle = |\psi\rangle \\ 2.\ \ U_1 U_2 \,|\psi\rangle = \mathrm{e}^{\mathrm{i}\theta}\, U_2 U_1 \,|\psi\rangle = \mathrm{e}^{\mathrm{i}\theta}\,|\psi\rangle \end{array}\right\} \textit{Contradicts assumption}$$

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Projective non-invertible symmetries <u>are compatible</u> with SPTs

➤ Loophole: symmetry operator has zero-eigenvalues

➤ $U R_{\mathsf{E}} = (-1)^L R_{\mathsf{E}} U \implies R_{\mathsf{E}} |\mathrm{GS}_{\mathrm{SPT}}\rangle = 0$ when L is odd

Projective $Z(G) \times \mathsf{Rep}(G)$ symmetry

The projective $\mathbb{Z}_2 \times \mathsf{Rep}(D_8)$ symmetry is a special case of a more general projective $Z(G) \times \mathsf{Rep}(G)$ symmetry

- $Z(G)$ is the center of a finite group G
- $\mathsf{Rep}(G)$ is the fusion category of representations of G

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$Z(G)$ symmetry operator U_z, with $z \in Z(G)$, satisfies

$$U_{z_1} U_{z_2} = U_{z_1 z_2}$$

$\mathsf{Rep}(G)$ symmetry operator R_Γ, with Γ an irrep of G, satisfies

$$R_{\Gamma_a} \times R_{\Gamma_b} = R_{\Gamma_a \otimes \Gamma_b} = R_{\oplus_c N_{ab}^c \Gamma_c} = \sum_c N_{ab}^c R_{\Gamma_c}$$

- Non-invertible symmetry when G is non-Abelian

Projective $Z(G) \times \mathsf{Rep}(G)$ symmetry

The projectivity arises through the relation

$$R_\Gamma U_z = (\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)})^L\, U_z R_\Gamma \quad \text{with } \mathrm{e}^{\mathrm{i}\phi_\Gamma(z)} = \mathrm{Tr}[\Gamma(z)]\,/\,d_\Gamma$$

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e.g., $\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)}$ when $G = \mathbb{Z}_2$ $(Z(\mathbb{Z}_2) = \mathbb{Z}_2)$

$z \backslash \Gamma$	$\mathbf{1}$	sign
$+1$	$+1$	$+1$
-1	$+1$	-1

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e.g., $\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)}$ when $G = D_8$ $(Z(D_8) = \mathbb{Z}_2)$

$z \backslash \Gamma$	$\mathbf{1}$	$\mathbf{1}_1$	$\mathbf{1}_2$	$\mathbf{1}_3$	E
$+1$	$+1$	$+1$	$+1$	$+1$	$+1$
-1	$+1$	$+1$	$+1$	$+1$	-1

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-1	$+1$	$+1$	$+1$	$+1$	-1

Explicit expressions of U_z and R_Γ for the Hilbert space $\bigotimes_j \mathbb{C}^{|G|}$

$$U_z = \sum_{\{g_j\}} |zg_1, \cdots, zg_L\rangle\langle g_1, \cdots, g_L| \qquad R_\Gamma = \sum_{\{g_j\}} \mathrm{Tr}[\Gamma(g_1 \cdots g_L)]\,|g_1, \cdots, g_L\rangle\langle g_1, \cdots, g_L|$$

(SPT)-LSM theorems

$$\boxed{R_\Gamma U_z = (\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)})^L\, U_z R_\Gamma}$$

There is an Lieb-Schultz-Mattis (LSM) theorem when $\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)}$ is non-trivial for a unitary R_Γ

[⋯; Matsui 2008; Chen, Gu, Wen 2010; Yao, Oshikawa 2020; Ogata, Tasaki 2021; Seifnashri 2023; Kapustin, Sopenko 2024]

- The LSM theorem forbids SPT phases

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- The LSM theorem forbids SPT phases

When there is no LSM theorem, the projective algebra gives rise to an SPT-LSM theorem

[Lu 2017; Yang, Jiang, Vishwanath, Ran 2017; Lu, Ran, Oshikawa 2017; ⋯]

- $R_\Gamma U_z = (\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)})^L\, U_z R_\Gamma$ forces any SPT state to satisfy $R_\Gamma\,|\mathrm{GS}\rangle = 0$ for nontrivial $(\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)})^L$
- Any SPT state must have non-zero entanglement

SPT-LSM theorem

To prove this SPT-LSM theorem, we

1. Use that the $Z(G)$ symmetry is on-site:

$$U_z = \prod_j U_j^{(z)} \quad \text{which satisfies} \quad R_\Gamma U_j^{(z)} = \mathrm{e}^{\mathrm{i}\phi_\Gamma(z)}\, U_j^{(z)} R_\Gamma$$

2. Assume that any unique gapped ground state $|\mathrm{GS}\rangle$ satisfies $R_\Gamma |\mathrm{GS}\rangle \neq 0$ for some $L = L^*$ $\quad \left(\mathrm{e}^{\mathrm{i}\phi_\Gamma(z) L^*} = 1\right)$

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Easy to prove assumption for product states in $\bigotimes_j \mathbb{C}^{|G|}$, *where*

$$R_\Gamma = \sum_{\{g_j\}} \mathrm{Tr}[\Gamma(g_1 \cdots g_L)]\,|g_1, \cdots, g_L\rangle\langle g_1, \cdots, g_L|$$

but it is true as long as there is an IR TQFT description

SPT-LSM theorem

If there is a unique gapped $|\mathrm{GS}\rangle$ that is a product state:

- $U_z|\mathrm{GS}\rangle = |\mathrm{GS}\rangle \implies U_j^{(z)}|\mathrm{GS}\rangle = |\mathrm{GS}\rangle$

Using the assumption, $R_\Gamma|\mathrm{GS}\rangle = \lambda_\Gamma|\mathrm{GS}\rangle$ at $L = L^*$:

$$\left.\begin{array}{l} 1.\ \ R_\Gamma U_j^{(z)}|\mathrm{GS}\rangle = R_\Gamma|\mathrm{GS}\rangle = \lambda_\Gamma|\mathrm{GS}\rangle \\ \\ 2.\ \ R_\Gamma U_j^{(z)}|\mathrm{GS}\rangle = \mathrm{e}^{\mathrm{i}\phi_\Gamma(z)}U_j^{(z)}R_\Gamma|\mathrm{GS}\rangle = \lambda_\Gamma \mathrm{e}^{\mathrm{i}\phi_\Gamma(z)}|\mathrm{GS}\rangle \end{array}\right\} \mathit{Contradiction}$$

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$\implies$ Cannot be an SPT state that is a product state at $L = L^*$

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$\implies$ Cannot be an SPT state that is a product state at $L = L^*$

$\implies$ By locality, there cannot be an SPT state that is a product state for any L

SPT-LSM theorem

If there is a unique gapped $|\mathrm{GS}\rangle$ that is a product state:

Therefore, the projective non-invertible symmetry prevents a product state SPT

- All SPTs must have non-zero entanglement

We argue that the projectivity always causes translation defects to carry nontrivial $\mathsf{Rep}(G)$ charge in SPT states

$\Longrightarrow$ By locality, there cannot be an SPT state that is a product state for any L

Outlook

We found a new class of entangled weak SPTs characterized by a projective $Z(G) \times \mathsf{Rep}(G)$ non-invertible symmetry

1. An exactly solvable model in a weak SPT phase characterized by a projective $\mathbb{Z}_2 \times \mathsf{Rep}(D_8)$ symmetry

2. General discussion on projective $Z(G) \times \mathsf{Rep}(G)$ weak SPTs $\implies$ an SPT-LSM theorem

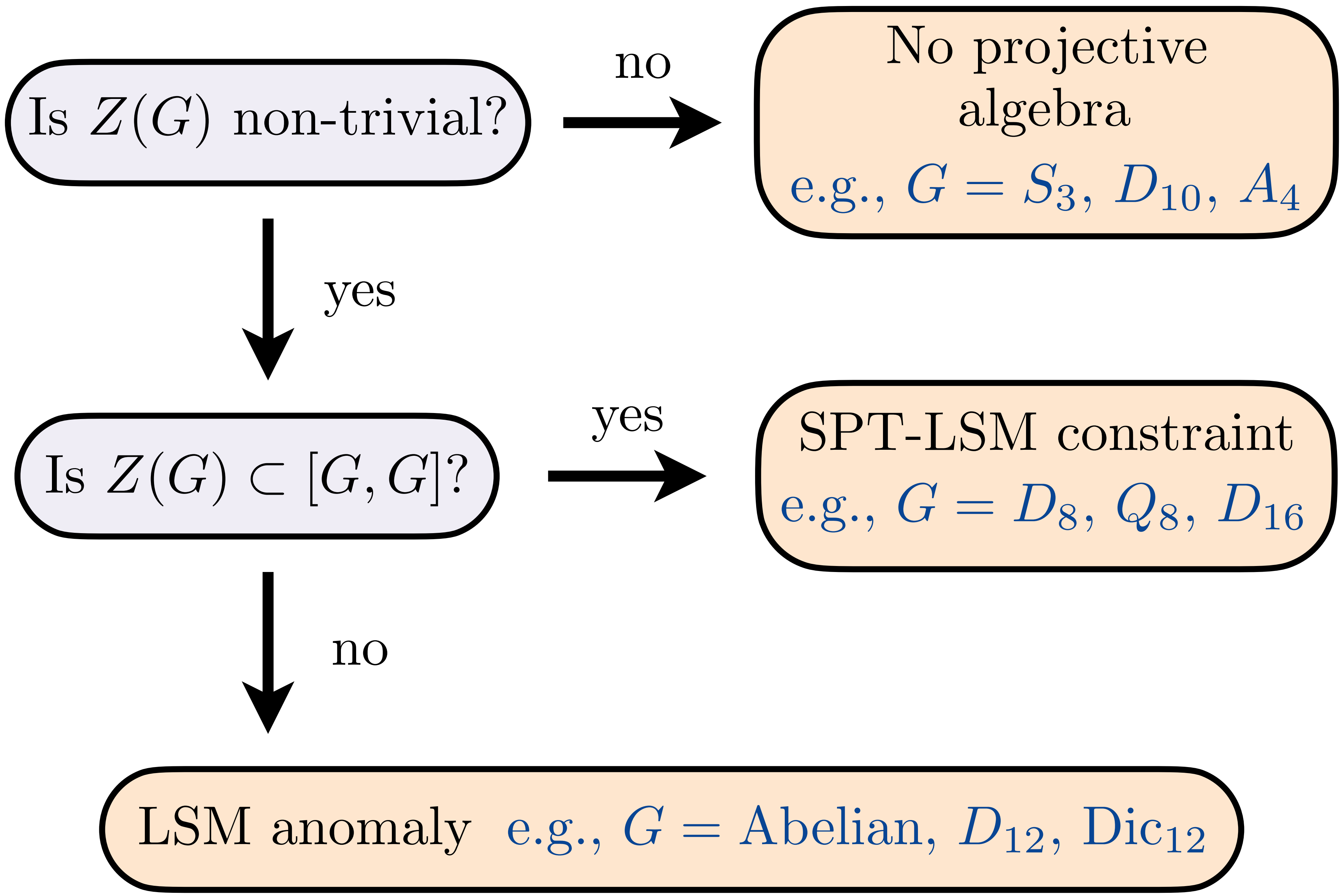
New quantum phases and models can be discovered using generalized symmetries as a guide!

SP, Lam, Aksoy arXiv:2409.18113

Back-up slides

(SPT)-LSM theorems

Whether there is an (SPT)-LSM theorem depends on G:



Non-invertible weak SPT

If there is an SPT phase, $R_\Gamma U_z = (\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)})^L \, U_z R_\Gamma$ forces its ground state to satisfy $R_\Gamma |\mathrm{GS}\rangle = 0$ for nontrivial $(\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)})^L$

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Two possibilities:

1. An SPT state satisfies $R_\Gamma|\mathrm{GS}\rangle = 0$ for all system sizes L
2. For $L = L^*$ where all $(\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)})^{L^*} = 1$, an SPT state satisfies $R_\Gamma|\mathrm{GS}\rangle = \lambda_\Gamma|\mathrm{GS}\rangle$, but $R_\Gamma|\mathrm{GS}\rangle = 0$ for $L \neq L^*$

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The first is incompatible with $1+1$D TQFT, where $\langle R_\Gamma\rangle = d_\Gamma$

[Chang, Lin, Shao, Wang, Yin 2018]

➤ Reasonable to assume that this SPT state at some $L = L^*$ is described by a TQFT in the IR

Non-invertible weak SPT

If there is an SPT phase, $R_\Gamma U_z = (\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)})^L U_z R_\Gamma$ forces its ground state to satisfy $R_\Gamma|\mathrm{GS}\rangle = 0$ for nontrivial $(\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)})^L$

Tw

1.

2.

At $L = L^*$, SPTs satisfy $R_\Gamma|\mathrm{GS}\rangle = \lambda_\Gamma|\mathrm{GS}\rangle$

At $L = L^* + 1$, SPTs satisfy $R_\Gamma|\mathrm{GS}\rangle = 0$

➤ All SPT states have translation defects dressed by non-trivial $\mathsf{Rep}(G)$ symmetry charge

➤ $\nexists$ a trivial SPT $\Longrightarrow$ SPT-LSM theorem

The $= d_\Gamma$

[Chang, Lin, Shao, Wang, Yin 2018]

➤ Reasonable to assume that this SPT state at some $L = L^*$ is described by a TQFT in the IR

D_8 fun facts

Dihedral group of order 8 $D_8 \simeq \left\langle r, s \mid r^2 = s^4 = 1, rsr = s^3 \right\rangle$

- Four 1d reps $\mathbf{1}$, $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3 = \mathbf{P}_1 \otimes \mathbf{P}_2$ and one 2d irrep E

$\mathbf{P}_i \otimes \mathbf{P}_i = \mathbf{1}$ $\quad$ $\mathsf{E} \otimes \mathsf{E} = \mathbf{1} \oplus \mathbf{P}_1 \oplus \mathbf{P}_2 \oplus \mathbf{P}_3$ $\quad$ $\mathsf{E} \otimes \mathbf{P}_i = \mathbf{P}_i \otimes \mathsf{E} = \mathsf{E}$

Projective $\mathbb{Z}_2 \times \mathsf{Rep}(D_8)$ bond algera

$$\mathfrak{B}\left[\mathsf{Rep}(D_8)\times\mathbb{Z}_2\right] = \left\langle \sigma_j^z,\ Z_j^2,\ Z_j\,Z_{j+1},\ \sigma_j^x\,C_{j+1}\,\sigma_{j+1}^x,\ X_j^{\sigma_j^z}\,X_{j+1}^\dagger \right\rangle$$

LSM anomaly in the XY model

Many-qubit model on a periodic chain with Hamiltonian

$$H = \sum_{j=1}^{L} J\,\sigma_j^x\,\sigma_{j+1}^x + K\,\sigma_j^y\,\sigma_{j+1}^y$$

- There is an LSM anomaly involving the $\mathbb{Z}_2^x \times \mathbb{Z}_2^y \times \mathbb{Z}_L$ symmetry [Chen, Gu, Wen 2010; Ogata, Tasaki 2021]

$$U_x = \prod_j \sigma_j^x, \quad U_y = \prod_j \sigma_j^y, \quad \text{and lattice translations } T$$

- Manifests through the projective algebras [Cheng, Seiberg 2023]

Translation defects	$\mathbb{Z}_2^x$ *defect*	$\mathbb{Z}_2^y$ *defect*
$U_xU_y = (-1)^L U_yU_x$	$U_y T = -\,T\,U_y$	$T\,U_x = -\,U_x T$

GROUP BASED QUDITS

A *G*-qudit is a $|G|$-level quantum mechanical system whose states are $|g\rangle$ with $g \in G$

- G is a finite group, e.g. $\mathbb{Z}_2$, S_3, D_8, SmallGroup(32,49)

Group based Pauli operators [Brell 2014]

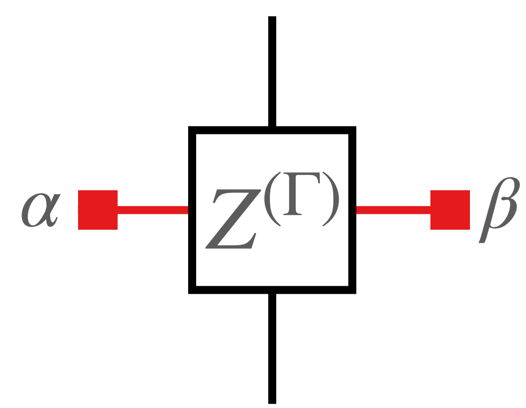
- X operators labeled by group elements

$\bar{g} \equiv g^{-1}$

$$\overrightarrow{X}^{(g)} = \sum_h |gh\rangle\langle h| \qquad \overleftarrow{X}^{(g)} = \sum_h |h\bar{g}\rangle\langle h|$$

- Z operators are MPOs labeled by irreps $\Gamma : G \to \mathrm{GL}(d_\Gamma, \mathbb{C})$

$$[Z^{(\Gamma)}]_{\alpha\beta} = \sum_h [\Gamma(h)]_{\alpha\beta} |h\rangle\langle h| \equiv$$



$(\alpha, \beta = 1,2,\cdots, d_\Gamma)$

GROUP BASED QUDITS

Example: $G = \mathbb{Z}_2$ where $g \in \{1, -1\}$ and $\Gamma \in \{\mathbf{1}, \mathbf{1}'\}$

$$\overrightarrow{X}^{(1)} = \overleftarrow{X}^{(1)} = [Z^{(\mathbf{1})}]_{11} = 1$$

$$\overrightarrow{X}^{(-1)} = \overleftarrow{X}^{(-1)} = \sigma^x \qquad\qquad [Z^{(\mathbf{1}')}]_{11} = \sigma^z$$

Group based Pauli operators satisfy

1. $\overrightarrow{X}^{(g)}\,\overrightarrow{X}^{(h)} = \overrightarrow{X}^{(gh)}$, $\overleftarrow{X}^{(g)}\,\overleftarrow{X}^{(h)} = \overleftarrow{X}^{(gh)}$, and $\overrightarrow{X}^{(g)}\,\overleftarrow{X}^{(h)} = \overleftarrow{X}^{(h)}\,\overrightarrow{X}^{(g)}$
2. $\overrightarrow{X}^{(g)}\,\overrightarrow{X}^{(h)} = \overrightarrow{X}^{(h)}\,\overrightarrow{X}^{(g)}$ iff g and h commute
3. $\overrightarrow{X}^{(g)}\,[Z^{(\Gamma)}]_{\alpha\beta} = [\Gamma(\bar{g})]_{\alpha\gamma}[Z^{(\Gamma)}]_{\gamma\beta}\,\overrightarrow{X}^{(g)}$
4. Unitarity: $\overrightarrow{X}^{(g)\dagger} = \overrightarrow{X}^{(\bar{g})}$, $\overleftarrow{X}^{(g)\dagger} = \overleftarrow{X}^{(\bar{g})}$, $[Z^{(\Gamma)\dagger}\,Z^{(\Gamma)}]_{\alpha\beta} = \delta_{\alpha\beta}$

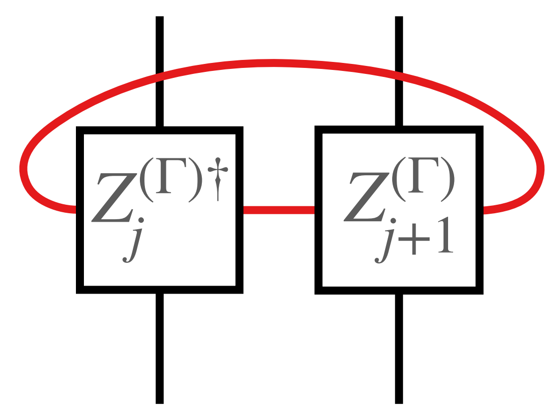
GROUP BASED XY MODEL

Group based Pauli operators are useful for constructing quantum lattice models [Brell 2014; Albert *et. al.* 2021; Fechisin, Tantivasadakarn, Albert 2023]

Group based *XY* model: Consider a periodic 1d lattice of L sites. On each site j resides a G-qudit and its Hamiltonian

$$H_{XY} = \sum_{j=1}^{L} \left(\sum_{\Gamma} J_\Gamma \, \mathrm{Tr}\left(Z_j^{(\Gamma)\dagger} Z_{j+1}^{(\Gamma)} \right) + \sum_g K_g \, \overleftarrow{X}_j^{(g)} \, \overrightarrow{X}_{j+1}^{(g)} \right) + \mathrm{hc}$$

$$\mathrm{Tr}\left(Z_j^{(\Gamma)\dagger} Z_{j+1}^{(\Gamma)} \right) = \sum_{\{g\}} \chi_\Gamma(\bar{g}_j g_{j+1}) \, | \{g\} \rangle \langle \{g\} | \equiv$$



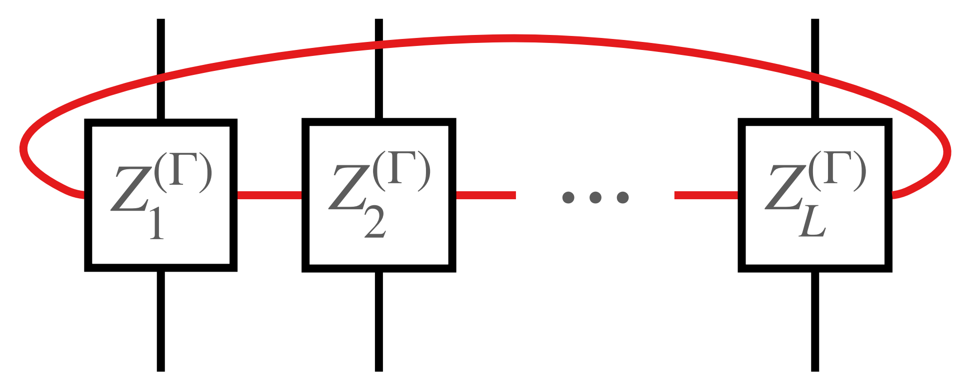
- For $G = \mathbb{Z}_2$, this is the ordinary quantum *XY* model

SYMMETRY OPERATORS

$$H_{XY} = \sum_{j=1}^{L} \left(\sum_{\Gamma} J_{\Gamma} \, \mathrm{Tr} \left(Z_j^{(\Gamma)\dagger} Z_{j+1}^{(\Gamma)} \right) + \sum_{g} K_g \, \overleftarrow{X}_j^{(g)} \, \overrightarrow{X}_{j+1}^{(g)} \right) + \mathrm{hc}$$

$\mathbb{Z}_L$ lattice translations: $T \mathcal{O}_j T^\dagger = \mathcal{O}_{j+1}$

Various internal symmetries:

- $Z(G)$ symmetry $U_z = \prod_j \overrightarrow{X}_j^{(z)}$ with $z \in Z(G)$
- Rep(G) symmetry $R_\Gamma = \mathrm{Tr} \left(\prod_{j=1}^{L} Z_j^{(\Gamma)} \right) \equiv$ 

$$R_{\Gamma_a} \times R_{\Gamma_b} = R_{\Gamma_a \otimes \Gamma_b} = R_{\oplus_c N_{ab}^c \Gamma_c} = \sum_c N_{ab}^c R_{\Gamma_c}$$

PROJECTIVE ALGEBRA FROM DEFECTS

$$U_z = \prod_j \vec{X}_j^{(z)} \qquad R_\Gamma = \mathrm{Tr}\left(\prod_{j=1}^{L} Z_j^{(\Gamma)}\right)$$

$$T_{\mathrm{tw}}^{(z)} = \vec{X}_I^{(z)}\, T \qquad T_{\mathrm{tw}}^{(\Gamma)} = \hat{Z}_I^{(\Gamma)}\,(T \otimes \mathbf{1})$$

Letting $\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)} \equiv \chi_\Gamma(z)/d_\Gamma$

Translation defects	$z \in Z(G)$ *defect*	$\Gamma \in \mathsf{Rep}(G)$ *defect*
$R_\Gamma U_z = (\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)})^L\, U_z R_\Gamma$	$R_\Gamma T_{\mathrm{tw}}^{(z)} = \mathrm{e}^{\mathrm{i}\phi_\Gamma(z)} T_{\mathrm{tw}}^{(z)} R_\Gamma$	$T_{\mathrm{tw}}^{(\Gamma)}\, U_z = \mathrm{e}^{\mathrm{i}\phi_\Gamma(z)} U_z\, T_{\mathrm{tw}}^{(\Gamma)}$

- Generalizes the $G = \mathbb{Z}_2$ projective algebra of the ordinary quantum XY model

GAUGING WEB

non-Abliean dipole symmetry

Non-invertible dipole symmetry

$$\begin{tikzcd}
 & \mathsf{Rep}(G) \times Z(G) \times \mathbb{Z}_L \text{ with projective algebra} \arrow[dl, bend right, "\mathsf{Rep}(G)"'] \arrow[dr, bend left, "Z(G)"] & \\
(G \times Z(G)) \rtimes \mathbb{Z}_L \arrow[dr, bend right, "Z(G)"'] & & (\mathsf{Rep}(G) \times Z(G)) \rtimes \mathbb{Z}_L \arrow[dl, bend left, "\mathsf{Rep}(G)"] \\
 & G \times Z(G) \times \text{non-invertible translations} &
\end{tikzcd}$$