Lattice T-duality from non-invertible symmetries in quantum spin chains

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Dualities in quantum systems

Dualities are maps between two seemingly distinct theories that are "secretly the same."

- Both conceptually and practically useful

Dualities in quantum systems

Dualities are maps between two seemingly distinct theories that are "secretly the same."

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Ask people on the street their favorite duality and hear:

T-duality, Particle-Vortex duality,

Kramers-Wannier duality

- These are not all the same notion of duality!
- Need to be more precise with "secretly the same."

Three* types of dualities

1. Exact duality: relates two different presentations of the same quantum system (is an isomorphism).

   - T-duality, Electromagnetic duality, Level-rank duality

** there are certainly more than just three*

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2. IR duality: relates two quantum systems that are distinct in the UV but the same in the IR.
   - Particle-vortex duality, Seiberg duality

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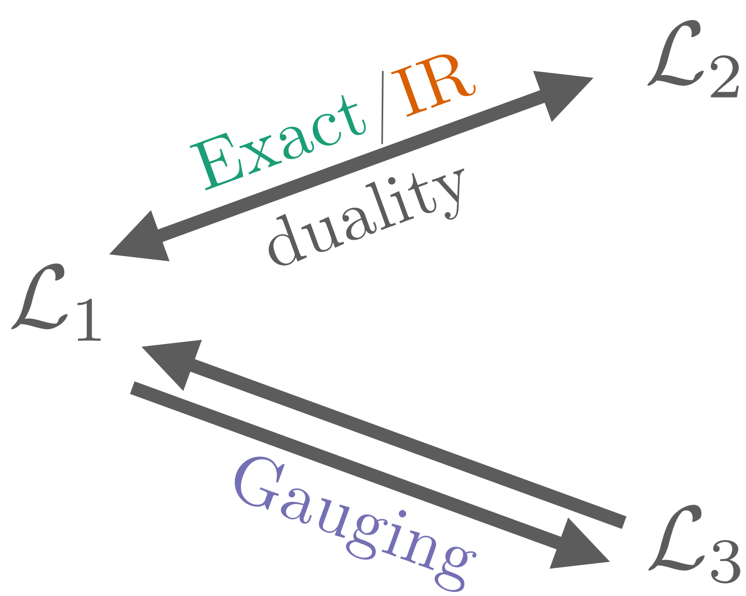
2. IR duality: relates two quantum systems that are distinct in the UV but the same in the IR.
   - Particle-vortex duality, Seiberg duality
3. Discrete gauging: relates two distinct quantum systems by gauging a discrete symmetry.
   - Kramers-Wannier duality, bosonization, fermionization

* *there are certainly more than just three*

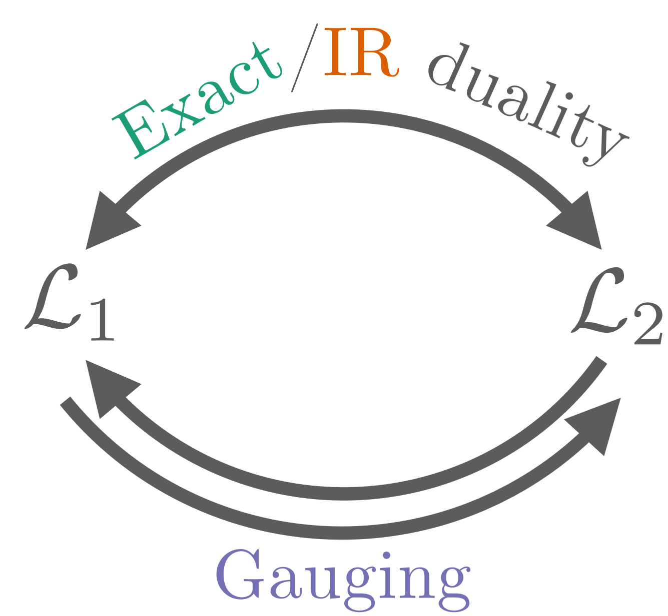
Three* types of dualities

Discrete gauging is generally unrelated to exact duality and IR duality!

Typical Scenario



Special Scenario



- Exact/IR dualities and discrete gauging always implement different maps

* *there are certainly more than just three*

T-duality in the compact boson CFT

The compact boson CFT at radius R is a $1+1$D CFT with

$$\mathcal{L}_R = \frac{R^2}{4\pi}\partial_\mu \Phi \partial^\mu \Phi, \qquad\qquad \Phi \sim \Phi + 2\pi$$

- T-duality: $\mathcal{L}_R$ and $\mathcal{L}_{1/R}$ are the same CFT:

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➤ T-duality: $\mathcal{L}_R$ and $\mathcal{L}_{1/R}$ are the same CFT:

$$\mathcal{Z} = \int \mathcal{D}\Theta\mathcal{D}W_\mu \mathrm{e}^{-\int \frac{R^2}{4\pi}W_\mu W^\mu - \frac{\mathrm{i}}{2\pi}\epsilon^{\mu\upsilon}\partial_\mu\Theta W_\upsilon}$$

Integrate out Θ

$$\implies \epsilon^{\mu\nu}\partial_\mu W_\nu = 0$$

$$\implies W_\mu = \partial_\mu\Phi$$

Integrate out W_μ

$$\implies W_\mu = -\frac{\mathrm{i}}{R^2}\epsilon_{\mu\nu}\partial^\nu\Theta$$

$$\mathcal{Z} = \int \mathcal{D}\Phi\, \mathrm{e}^{-\int \mathcal{L}_R}$$

$$\mathcal{Z} = \int \mathcal{D}\Theta\, \mathrm{e}^{-\int \mathcal{L}_{1/R}}$$

T-duality in the compact boson CFT

T-duality is an isomorphism of *all* operators & *all* states of $\mathcal{L}_R$ and $\mathcal{L}_{1/R}$ (it is an exact duality). Preserves symmetries!

- Example: The CFT has $\mathrm{U}(1)^{\mathcal{M}}$ momentum and $\mathrm{U}(1)^{\mathcal{W}}$ winding symmetries

$$J^{\mathcal{M}}_\mu = \frac{R^2}{2\pi}\partial_\mu \Phi$$

$$J^{\mathcal{W}}_\mu = \frac{1}{2\pi}\epsilon_{\mu\nu}\partial^\nu \Phi$$

$$\begin{array}{ccc} \mathcal{L}_R & & \mathcal{L}_{1/R} \\ J^{\mathcal{M}} & \xrightarrow[\text{map}]{\text{T-duality}} & J^{\mathcal{W}} \\ J^{\mathcal{W}} & & J^{\mathcal{M}} \\ \vdots & & \vdots \end{array}$$

- Exchange of momentum and winding symmetries is a hallmark of T-duality!

Gauging in the compact boson CFT

The $\mathrm{U}(1)^{\mathcal{M}}$ symmetry transformation $\Phi \to \Phi + C$

- Gauging $\mathbb{Z}_N^{\mathcal{M}} \subset \mathrm{U}(1)^{\mathcal{M}}$ causes $\Phi \sim \Phi + 2\pi/N$, which is equivalent to $\Phi \sim \Phi + 2\pi$ with $R \to R/N$

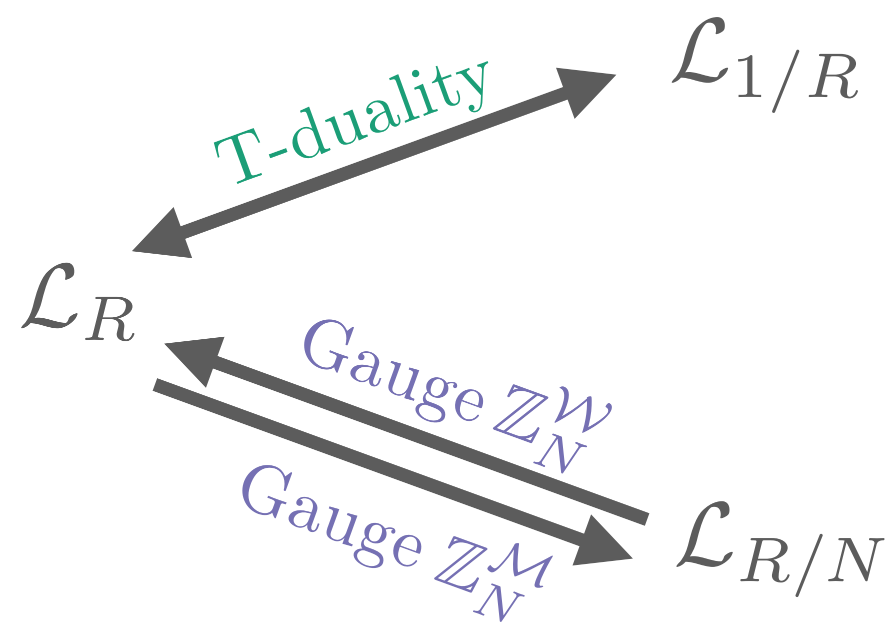
$$\begin{array}{ccc} \mathcal{L}_R & & \mathcal{L}_{R/N} \\ J^{\mathcal{M}} & \xrightarrow[\mathbb{Z}_N^{\mathcal{M}}]{\text{Gauge}} & N J^{\mathcal{M}} \\ J^{\mathcal{W}} & & J^{\mathcal{W}}/N \\ \vdots & & \vdots \end{array}$$

- Has a nontrivial kernel spanned by $Q^{\mathcal{M}} \notin N\mathbb{Z}$ states

Gauging maps are non-invertible (not bijective)!

Non-invertible symmetry at $R = \sqrt{N}$

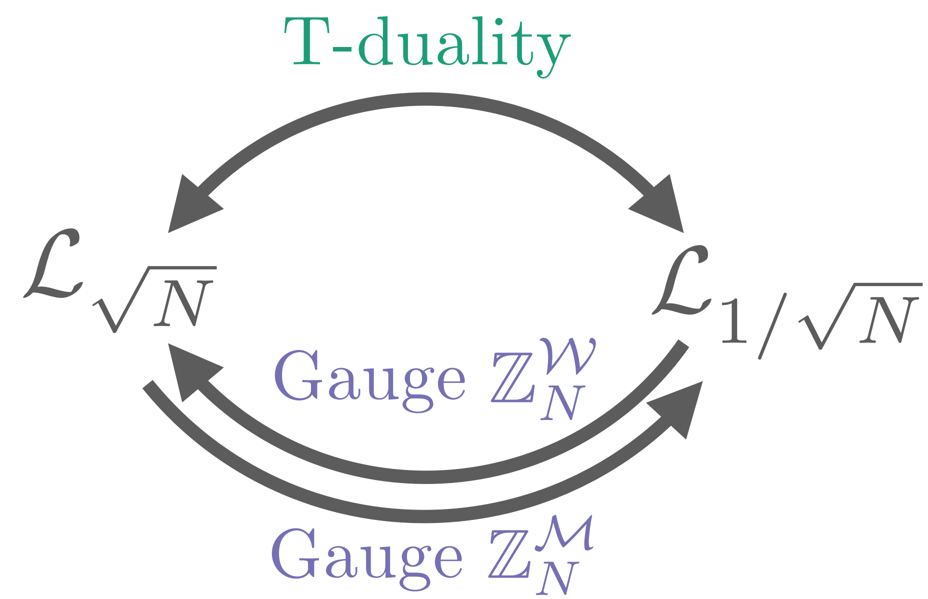
When $R \neq \sqrt{N}$, the image of $\mathcal{L}_R$ under T-duality and Gauging $\mathbb{Z}_N^{\mathcal{M}}$ is <u>different</u>.



Non-invertible symmetry at $R = \sqrt{N}$

When $R = \sqrt{N}$, the image of $\mathcal{L}_R$ under T-duality and Gauging $\mathbb{Z}_N^{\mathcal{M}}$ is <u>the same</u>.

➤ T-duality: Isomorphism of $\mathcal{L}_{\sqrt{N}}$ and its $\mathbb{Z}_N^{\mathcal{M}}$-gauged theory



Non-invertible symmetry at $R = \sqrt{N}$

When $R = \sqrt{N}$, the image of $\mathcal{L}_R$ under T-duality and Gauging $\mathbb{Z}_N^{\mathcal{M}}$ is the same.

$$\begin{tikzcd} \mathcal{L}_{\sqrt{N}} \arrow[r, bend left, leftrightarrow, "\text{T-duality}"] \arrow[r, bend right, "\text{Gauge } \mathbb{Z}_N^{\mathcal{M}}"'] & \mathcal{L}_{1/\sqrt{N}} \arrow[l, bend left=20, "\text{Gauge } \mathbb{Z}_N^{\mathcal{W}}"'] \end{tikzcd}$$

- T-duality: Isomorphism of $\mathcal{L}_{\sqrt{N}}$ and its $\mathbb{Z}_N^{\mathcal{M}}$-gauged theory

- Non-invertible symmetry [Thorngren, Wang '21; Choi, Córdova, Hsin, Lam, Shao '21]

$$\begin{array}{ccccc} \mathcal{L}_{\sqrt{N}} & & \mathcal{L}_{1/\sqrt{N}} & & \mathcal{L}_{\sqrt{N}} \\ J^{\mathcal{M}} & \xrightarrow[\mathbb{Z}_N^{\mathcal{M}}]{\text{Gauge}} & N J^{\mathcal{M}} & \xrightarrow[\text{map}]{\text{T-duality}} & N J^{\mathcal{W}} \\ J^{\mathcal{W}} & & J^{\mathcal{W}}/N & & J^{\mathcal{M}}/N \\ \vdots & & \vdots & & \vdots \end{array}$$

Non-invertible symmetry at $R = \sqrt{N}$

When $R = \sqrt{N}$, the image of $\mathcal{L}_R$ T-duality

un

is t $\sqrt{N}$

➤

and its $\mathbb{Z}_N^{\mathcal{M}}$-gauged theory Gauge $\mathbb{Z}_N^{\mathcal{M}}$

The existence of U(1) momentum, U(1) winding, & this non-invertible symmetry provides an invariant *definition* of T-duality.

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T-duality and qubits?

Can T-duality exist in lattice models that flow to the compact boson in the IR? How about in qubit models?

<u>This talk</u>

1. In the XX model, there is a non-invertible symmetry and corresponding lattice T-duality
2. Encounter a U(1) lattice winding symmetry and conserved charges forming the Onsager algebra. We'll discuss 't Hooft anomalies and prove a gaplessness constraint
3. Explore symmetric deformations of the XX model

The XX model

Consider $1+1$ D quantum lattice model on a finite ring with a qubit residing on each site j

- The number of sites L is even
- Pauli operators satisfy $X_{j+L} = X_j$ and $Z_{j+L} = Z_j$

XX model Hamiltonian [Lieb, Schultz, Mattis '61; Baxter '71; ⋯]

$$H_{\mathrm{XX}} = \sum_{j=1}^{L} (X_j X_{j+1} + Y_j Y_{j+1})$$

- Spin rotation $\mathrm{U}(1)^{\mathrm{M}}$ symmetry

$$Q^{\mathrm{M}} = \frac{1}{2} \sum_{j=1}^{L} Z_j$$

The XX model

$$H_{\mathrm{XX}} = 2\sum_{j=1}^{L}\left(\sigma_j^+\sigma_{j+1}^- + \sigma_j^-\sigma_{j+1}^+\right) \qquad \sigma_j^\pm = \frac{1}{2}\left(X_j \pm \mathrm{i}Y_j\right)$$

➤ $\mathrm{e}^{\mathrm{i}\phi Q^{\mathrm{M}}}$ transforms $\sigma_j^\pm \to \mathrm{e}^{\pm\mathrm{i}\phi}\sigma_j^\pm$

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IR limit of the XX model

The IR of the XX model is described by the compact boson CFT at $R = \sqrt{2}$

[Alcaraz, Barber, Batchelor '87; Baake, Christe, Rittenberg '88]

- The IR limit: focus on low-energy states within an $\mathcal{O}(L^0)$ energy window above the ground state and take $L \to \infty$

$$
\begin{array}{ccc}
\sigma_j^+ & & \exp[\mathrm{i}\,\Phi] \\
Q^{\mathrm{M}} & \xrightarrow{\text{IR limit}} & \mathcal{Q}^{\mathcal{M}} = \int J_0^{\mathcal{M}} \\
\vdots & & \vdots
\end{array}
$$

- Q^{M} generates a U(1) momentum symmetry on the lattice

Gauging $\mathbb{Z}_2^{\mathrm{M}}$ in the XX model

Does the XX model have a lattice T-duality?

- In the IR: implements an isomorphism between the $R = \sqrt{2}$ compact boson CFT and its $\mathbb{Z}_2^{\mathcal{M}}$ gauged theory

Let's gauge the $\mathbb{Z}_2^{\mathrm{M}}$ symmetry $\mathrm{e}^{\mathrm{i}\pi Q^{\mathrm{M}}} = \prod_{j=1}^{L} (-1)^j Z_j$ in the XX model

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$$\begin{pmatrix} (-1)^j Z_j \\ X_j X_{j+1} \end{pmatrix} \xrightarrow{\text{Gauge } \mathbb{Z}_2^{\mathrm{M}}} \begin{pmatrix} Z_j Z_{j+1} \\ X_{j+1} \end{pmatrix}$$

➤ $\mathbb{Z}_2^{\mathrm{M}}$ gauged Hamiltonian

$$H_{\mathrm{XX}/\mathbb{Z}_2^{\mathrm{M}}} = \sum_{j=1}^{L} (X_j + Z_{j-1} X_j Z_{j+1})$$

Lattice T-duality

Hamiltonians are unitarily equivalent: $H_{\mathrm{XX}} = U_{\mathrm{T}} H_{\mathrm{XX}/\mathbb{Z}_2^{\mathrm{M}}} U_{\mathrm{T}}^{-1}$

$$U_{\mathrm{T}} = \prod_{n=1}^{L/2} \left(\mathrm{e}^{\mathrm{i}\frac{\pi}{4} Z_{2n+1}} \mathrm{e}^{\mathrm{i}\frac{\pi}{4} X_{2n+1}} \mathrm{e}^{-\mathrm{i}\frac{\pi}{4} X_{2n}} \mathrm{CZ}_{2n,2n+1} \right)$$

- U_{T} implements an isomorphism between the XX model and its $\mathbb{Z}_2^{\mathrm{M}}$ gauged theory

T-duality from the isomorphism pov!

What about the symmetry pov?

Non-invertible symmetry of the XX model

Hamiltonians are unitarily equivalent: $H_{\mathrm{XX}} = U_{\mathrm{T}} H_{\mathrm{XX}/\mathbb{Z}_2^{\mathrm{M}}} U_{\mathrm{T}}^{-1}$

➤ Implies there is a non-invertible symmetry D:

$$\begin{pmatrix} Z_{2n-1} \\ Z_{2n} \\ X_{2n-1}X_{2n} \\ X_{2n}X_{2n+1} \end{pmatrix} \xrightarrow{\text{Gauge } \mathbb{Z}_2^{\mathrm{M}}} \begin{pmatrix} -Z_{2n-1}Z_{2n} \\ Z_{2n}Z_{2n+1} \\ X_{2n} \\ X_{2n+1} \end{pmatrix} \xrightarrow{U_{\mathrm{T}}} \begin{pmatrix} X_{2n-1}Y_{2n} \\ -Y_{2n}X_{2n+1} \\ X_{2n}X_{2n+1} \\ Y_{2n}Y_{2n+1} \end{pmatrix}$$

➤ Symmetry because D: $H_{\mathrm{XX}} \to U_{\mathrm{T}} H_{\mathrm{XX}/\mathbb{Z}_2^{\mathrm{M}}} U_{\mathrm{T}}^{-1} = H_{\mathrm{XX}}$

➤ Non-invertible because $\mathsf{D}\mathrm{e}^{\mathrm{i}\pi Q^{\mathrm{M}}} = \mathsf{D}$

$$\implies \mathsf{D}|\psi\rangle = 0 \quad \text{if} \quad \mathrm{e}^{\mathrm{i}\pi Q^{\mathrm{M}}}|\psi\rangle = -|\psi\rangle$$

$$\implies \mathsf{D}^{-1} \text{ does not exist}$$

Lattice winding symmetry

What about the winding symmetry?

Lattice winding symmetry

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- Acting D on Q^{M}

$$\mathsf{D}Q^{\mathrm{M}} = 2Q^{\mathrm{W}}\mathsf{D}$$

$$\text{where } Q^{\mathrm{W}} = \frac{1}{4}\sum_{n=1}^{L/2}\left(X_{2n-1}Y_{2n} - Y_{2n}X_{2n+1}\right)$$

$\Longrightarrow$ There is a lattice winding charge*

- Acting D on Q^{W}

$$\mathsf{D}Q^{\mathrm{W}} = \frac{1}{2}Q^{\mathrm{M}}\mathsf{D}$$

* *Known conserved charge of the XX model [Vernier, O'Brien, Fendley '18; Miao '21; Popkov, Zhang, Göhmann, Klümper '23]*

Lattice winding symmetry

What about the winding symmetry?

The XX model has a lattice T-duality

- Isomorphism between H_{XX} and $H_{\mathrm{XX}/\mathbb{Z}_2^{\mathrm{M}}}$
- Conserved lattice Q^{M} and Q^{W} charges
- Non-invertible symmetry exchanging Q^{M} and Q^{W}

$$\mathsf{D}Q^{\mathrm{M}} = 2Q^{\mathrm{W}}\mathsf{D} \qquad \mathsf{D}Q^{\mathrm{W}} = \frac{1}{2}Q^{\mathrm{M}}\mathsf{D}$$

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Onsager algebra

The charges Q^{M} and Q^{W} do not commute on the lattice

$$[Q^{\mathrm{M}}, Q^{\mathrm{W}}] \neq 0 \xrightarrow{\text{IR limit}} [\mathcal{Q}^{\mathcal{M}}, \mathcal{Q}^{\mathcal{W}}] = 0$$

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- They generate the Onsager algebra. Formed by conserved charges Q_n, G_n , with $Q_0 = Q^{\mathrm{M}}$ and $Q_1 = 2Q^{\mathrm{W}}$, satisfying

[Onsager '44; Vernier, O'Brien, Fendley '18; Miao '21]

$$[Q_n, Q_m] = \mathrm{i}G_{m-n} \qquad\qquad [G_n, G_m] = 0$$

$$[Q_n, G_m] = 2\mathrm{i}\,(Q_{n-m} - Q_{n+m})$$

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$$[Q_n, G_m] = 2\mathrm{i}\left(Q_{n-m} - Q_{n+m}\right)$$

$$Q_n \xrightarrow{\text{IR limit}} \begin{cases} 2\mathcal{Q}^{\mathcal{W}} & n \text{ odd} \\ \mathcal{Q}^{\mathcal{M}} & n \text{ even} \end{cases} \qquad\qquad G_n \xrightarrow{\text{IR limit}} 0$$

Anomalies

While searching for lattice T-duality, we found symmetries of the XX model directly related to those in the IR.

- How do these symmetries match the 't Hooft anomalies — obstructions to gauging — in the IR?

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Consider the symmetry operators

$$\mathrm{e}^{\mathrm{i}\pi Q^{\mathrm{M}}} = \prod_{j=1}^{L} (-1)^j Z_j \qquad \mathrm{e}^{\mathrm{i}\theta Q^{\mathrm{W}}} \qquad C = \prod_{j=1}^{L} X_j$$

- Described by the group $\mathbb{Z}_2^{\mathrm{M}} \times \mathrm{U}(1)^{\mathrm{W}} \rtimes \mathbb{Z}_2^{\mathrm{C}}$
- Subgroup of the UV and IR symmetry groups

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Consi

$\mathbb{Z}_2^{\mathrm{C}} \times \mathbb{Z}_2^{\mathrm{M}} \times \mathbb{Z}_2^{\mathrm{W}}$
type III anomaly

$\mathbb{Z}_2^{\mathrm{C}}$

$\mathrm{U}(1)^{\mathrm{M}}$

$\mathrm{U}(1)^{\mathrm{W}}$

$\mathbb{Z}_2^{\mathrm{M}} \times \mathrm{U}(1)^{\mathrm{W}}$
mixed anomaly

- D
- Su

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Projective algebras

$\mathbb{Z}_2^{\mathrm{C}}$

$\mathrm{U}(1)^{\mathrm{M}}$

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$\mathbb{Z}_2^{\mathrm{C}}$

$\mathrm{U}(1)^{\mathrm{M}}$

$\mathrm{U}(1)^{\mathrm{W}}$

Spectral flow

$\mathbb{Z}_2^{\mathrm{M}} \times \mathrm{U}(1)^{\mathrm{W}}$

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Perturbative anomalies in the IR

The mixed anomaly of $\mathrm{U}(1)^{\mathcal{M}} \times \mathrm{U}(1)^{\mathcal{W}}$ in the compact boson CFT is a "perturbative anomaly"

- Cannot be matched by gapped phases $\Longrightarrow$ enforces gaplessness [⋯ ; Córdova, Freed, Teleman '24]

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Do the lattice momentum and winding symmetries enforce gaplessness?

- Does the Onsager algebra match the perturbative anomaly?

Answer: Yes! Can show by fermionizing the XX model

Fermionizing the XX model

We fermionize the XX model by gauging the $\mathbb{Z}_2^{\mathrm{M}}$ symmetry using complex fermion operators c_j and $c_j^\dagger$

[⋯; Radičević '18; Borla, Verresen, Shah, Moroz '20; Seiberg, Shao '23; Aksoy, Mudry, Furusaki, Tiwari '23; Seifnashri '23]

- In terms of real fermions $c_j = (a_j + \mathrm{i} b_j)/2$

$$\{a_j, b_{j'}\} = 0 \qquad \{a_j, a_{j'}\} = 2\delta_{j,j'} \qquad \{b_j, b_{j'}\} = 2\delta_{j,j'}$$

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Gauging implemented using the Gauss law

$$G_j = (-1)^j Z_j \mathrm{i} a_j b_j = 1$$

➤ Map to gauged theory summarized by

$$Z_j \to \mathrm{i} a_j b_j \qquad X_j X_{j+1} \to \begin{cases} -\mathrm{i} a_j a_{j+1} & j \text{ odd} \\ -\mathrm{i} b_j b_{j+1} & j \text{ even} \end{cases}$$

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[⋯; Radičević '18 … ashri '23]

$$H_{\mathrm{XX}} \xrightarrow{\text{Fermionize}} -\mathrm{i}\sum_{j=1}^{L}(a_j a_{j+1} + b_j b_{j+1})$$

$$Q^{\mathrm{M}} \xrightarrow{\text{Fermionize}} \frac{1}{2}\sum_{j=1}^{L}\mathrm{i}a_j b_j \equiv Q^{\mathrm{V}}$$

$$2Q^{\mathrm{W}} \xrightarrow{\text{Fermionize}} \frac{1}{2}\sum_{j=1}^{L}\mathrm{i}a_j b_{j+1} \equiv Q^{\mathrm{A}}$$

➤ In te…

$\{a_j, b$ … $2\delta_{j,j'}$

Gauging…

➤ Map…

$$Z_j \to \mathrm{i}a_j b_j \qquad X_j X_{j+1} \to \begin{cases} -\mathrm{i}a_j a_{j+1} & j \text{ odd} \\ -\mathrm{i}b_j b_{j+1} & j \text{ even} \end{cases}$$

Symmetric Q^{V} and Q^{A} Hamiltonians

We assume the Hamiltonian is local:

$$H_f = \sum_n \sum_{j=1}^{L} g_{j,n} H_j^{(n)}$$

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1. $e^{-\mathrm{i}\frac{\pi}{2}Q^{\mathrm{A}}} e^{\mathrm{i}\frac{\pi}{2}Q^{\mathrm{V}}} : (a_j, b_j) \to (a_{j-1}, b_{j+1})$ invariance requires $H_j^{(n)}$ to <u>not</u> have terms mixing a_j and b_j and $g_{j,n} = g_n$

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2. Under the $\mathrm{e}^{\mathrm{i}\phi Q^{\mathrm{V}}}$ transformation

$$a_j \to \cos(\phi)\, a_j + \sin(\phi)\, b_j \qquad b_j \to \cos(\phi)\, b_j - \sin(\phi)\, a_j$$

$\implies$ Only allowed $H_j^{(n)}$ are

$$H_j^{(n)} = \mathrm{i} a_j a_{j+n} + \mathrm{i} b_j b_{j+n}$$

Enforced gaplessness

$$H_f = \mathrm{i}\sum_n \sum_{j=1}^{L} g_n (a_j a_{j+n} + b_j b_{j+n})$$

The Q^{V} and Q^{A} symmetric Hamiltonians are always gapless

➤ In momentum space:

$$H_f = \sum_{k\in\mathrm{BZ}} \omega_k\, c_k^\dagger c_k, \qquad \omega_k = 4\sum_n g_n \sin(2\pi k n/L)$$

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Bosonization: one-to-one correspondence between H_f and qubit Hamiltonians commuting with Q^{M} and Q^{W}

- Bosonization maps implemented by gauging $(-1)^F$
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- In momentum space:

The perturbative anomaly of the compact boson CFT
is matched by the Onsager algebra

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Enforced gaplessness

$$H_f = \mathrm{i}\sum_n \sum_{j=1}^{L} g_n (a_j a_{j+n} + b_j b_{j+n})$$

When $L = 0 \bmod 4$, there is a unitary frame in which

$$Q^{\mathrm{M}} = -\frac{1}{2}\sum_{j=1}^{L} Z_j \qquad Q^{\mathrm{W}} = -\frac{1}{4}\sum_{j=1}^{L} X_j X_{j+1}$$

- Any qubit chain commuting with $\sum_j Z_j$ and $\sum_j X_j X_{j+1}$ is gapless

- Bosonization maps implemented by gauging $(-1)^F$
- Because H_f is gapless, Q^{M} and Q^{W} enforce gaplessness

Symmetric deformations

Can find $\mathrm{U}(1)^{\mathrm{M}}$ and $\mathrm{U}(1)^{\mathrm{W}}$ symmetric deformations of the XX model by bosonizing H_f

$$H_j^{(1)} \xrightarrow{\text{bosonize}} X_j X_{j+1} + Y_j Y_{j+1}$$

$$H_j^{(2)} \xrightarrow{\text{bosonize}} Y_j Z_{j+1} X_{j+2} - X_j Z_{j+1} Y_{j+2}$$

$$H_j^{(3)} \xrightarrow{\text{bosonize}} X_j Z_{j+1} Z_{j+2} X_{j+3} + Y_j Z_{j+1} Z_{j+2} Y_{j+3}$$

$$\vdots$$

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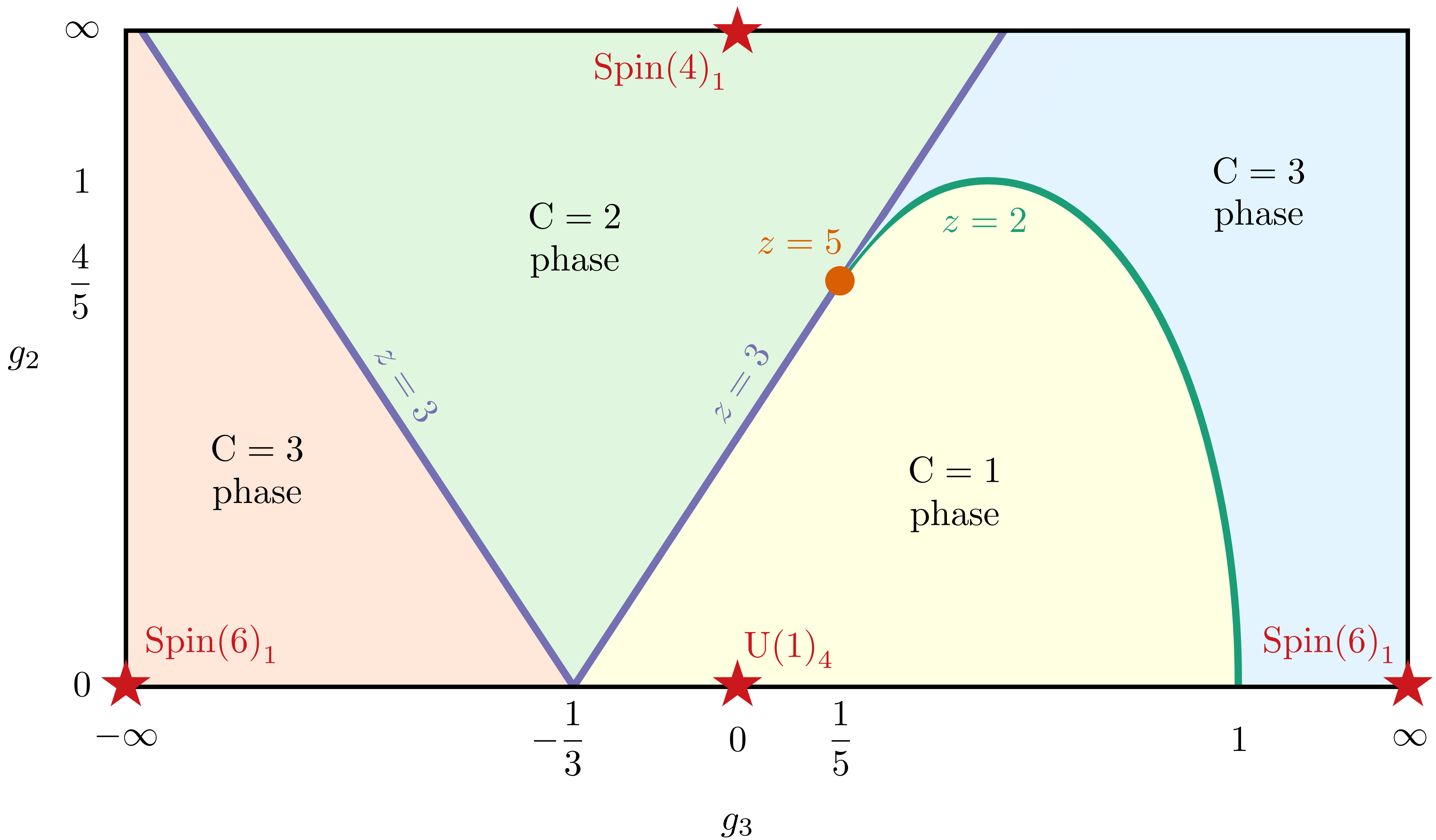
$$\vdots$$

Non-invertible symmetry D arises from $\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}Q^{\mathrm{V}}} T_a$

- $\mathrm{U}(1)^{\mathrm{M}}$ and $\mathrm{U}(1)^{\mathrm{W}}$ guarantee the non-invertible symmetry and a lattice T-duality

Simplest 2-parameter phase diagram

$$H\left(g_2, g_3\right) = H_{\mathrm{XX}} + \sum_{j=1}^{L}\left(g_2 H_j^{(2)} + g_3 H_j^{(3)}\right)$$



Recap and outlook

Many aspects of the compact boson CFT surprisingly exist exactly in the XX model

1. Lattice T-duality and non-invertible symmetry
2. Lattice winding symmetry and 't Hooft anomalies
3. Symmetric deformations of the XX model

Recap and outlook

Many aspects of the compact boson CFT surprisingly exist exactly in the XX model

1. Lattice T-duality and non-invertible symmetry
2. Lattice winding symmetry and 't Hooft anomalies
3. Symmetric deformations of the XX model

Tip of an iceberg?

1. T-duality for other radii? S-duality in $3+1$D qubit models?
2. General relationship between perturbative anomalies and algebras? Between exact dualities of QFTs and unitary transformations in quantum lattice models?

Back-up slides

Lattice T-duality

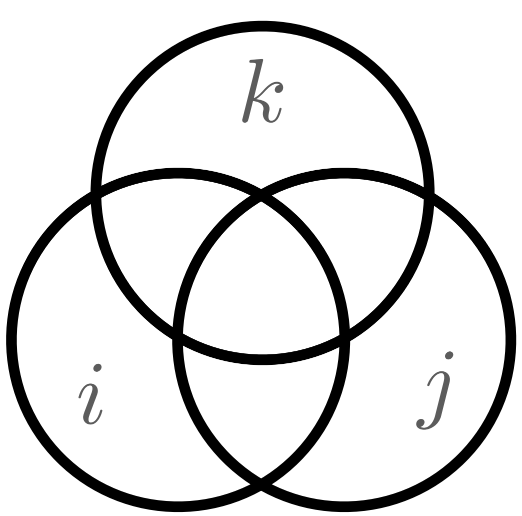
Can T-duality exist in lattice models that flow to the compact boson in the IR?

Yes: exists in the Modified Villain model

[Gross, Klebanov '90; Gorantla, Lam, Seiberg, Shao '21; Cheng, Seiberg '22; Fazza, Sulejmanpasic '22]

➤ Careful lattice regularization of the compact boson CFT

Patches $\{U_i\}$ $\quad \Phi_i : U_i \to \mathbb{R} \quad n_{ij} : U_i \cap U_j \to \mathbb{Z}$



➤ $\Phi_i - \Phi_j = 2\pi n_{ij}$ on $U_i \cap U_j$

➤ Gauge redundancy with $m_i \in \mathbb{Z}$

$$\Phi_i \sim \Phi_i + 2\pi m_i$$

$$n_{ij} \sim n_{ij} + m_i - m_j$$

Lattice T-duality

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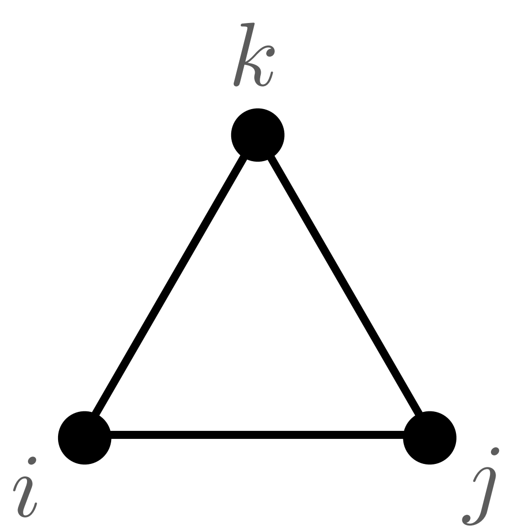
- Careful lattice regularization of the compact boson CFT

Spacetime lattice $\Phi_i \in \mathbb{R}$ $n_{ij} \in \mathbb{Z}$



- Gauge redundancy with $m_i \in \mathbb{Z}$

$$\Phi_i \sim \Phi_i + 2\pi m_i$$

$$n_{ij} \sim n_{ij} + m_i - m_j$$

- Infinite-dimensional local Hilbert space

Non-invertible symmetry action

Hamiltonians are unitarily equivalent: $H_{\mathrm{XX}} = U_{\mathrm{T}} H_{\mathrm{XX}/\mathbb{Z}_2^{\mathrm{M}}} U_{\mathrm{T}}^{-1}$

- There is a non-invertible symmetry operator D

$$\begin{pmatrix} Z_{2n-1} \\ Z_{2n} \\ X_{2n-1}X_{2n} \\ X_{2n}X_{2n+1} \end{pmatrix} \xrightarrow{\text{Gauge } \mathbb{Z}_2^{\mathrm{M}}} \begin{pmatrix} -Z_{2n-1}Z_{2n} \\ Z_{2n}Z_{2n+1} \\ X_{2n} \\ X_{2n+1} \end{pmatrix} \xrightarrow{U_{\mathrm{T}}} \begin{pmatrix} X_{2n-1}Y_{2n} \\ -Y_{2n}X_{2n+1} \\ X_{2n}X_{2n+1} \\ Y_{2n}Y_{2n+1} \end{pmatrix}$$

- Implies that

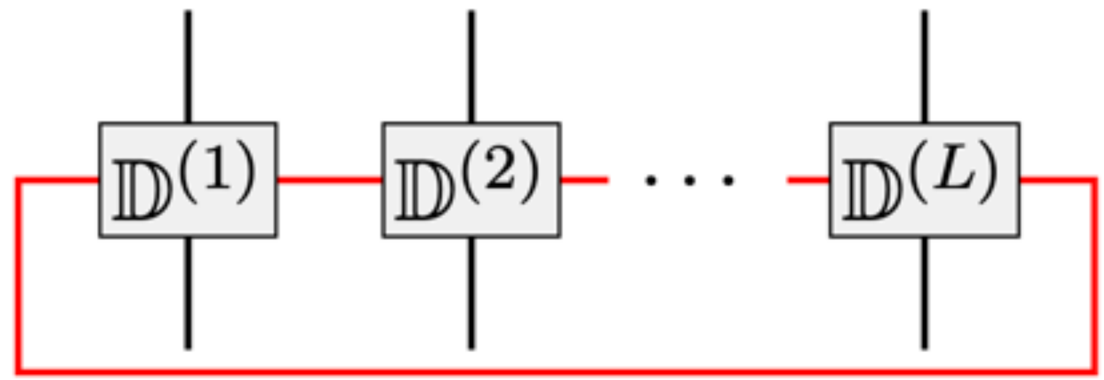
$$\mathsf{D} Z_j = \begin{cases} (X_j Y_{j+1})\,\mathsf{D} & j \text{ odd}\,, \\ (-Y_j X_{j+1})\,\mathsf{D} & j \text{ even}\,, \end{cases} \qquad \mathsf{D} X_j X_{j+1} = \begin{cases} (X_{j+1} X_{j+2})\,\mathsf{D} & j \text{ odd} \\ (Y_j Y_{j+1})\,\mathsf{D} & j \text{ even} \end{cases}$$

$$\mathsf{D}^2 = (1 + \mathrm{e}^{\mathrm{i}\pi Q^{\mathrm{M}}})\, T\, \mathrm{e}^{-\mathrm{i}\frac{\pi}{2} Q^{\mathrm{M}}}, \qquad \mathsf{D}\, \mathrm{e}^{\mathrm{i}\pi Q^{\mathrm{M}}} = \mathrm{e}^{\mathrm{i}\pi Q^{\mathrm{M}}} \mathsf{D} = \mathsf{D},$$

$$T \mathsf{D} T^{-1} = \mathrm{e}^{\mathrm{i}\frac{\pi}{2} Q^{\mathrm{M}}} \mathrm{e}^{\mathrm{i}\pi Q^{\mathrm{W}}} \mathsf{D}, \qquad \mathsf{D}^\dagger = \mathsf{D} T^{-1} \mathrm{e}^{\mathrm{i}\frac{\pi}{2} Q^{\mathrm{M}}}$$

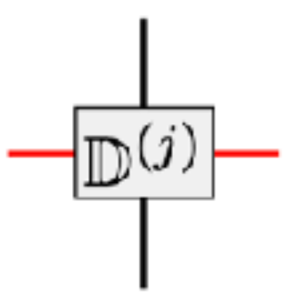
D as an Matrix Product Operator

$$\mathsf{D} = \mathrm{Tr}\left(\prod_{j=1}^{L} \mathbb{D}^{(j)}\right) \equiv$$



where the MPO tensor

$$\mathbb{D}^{(j)} \equiv$$

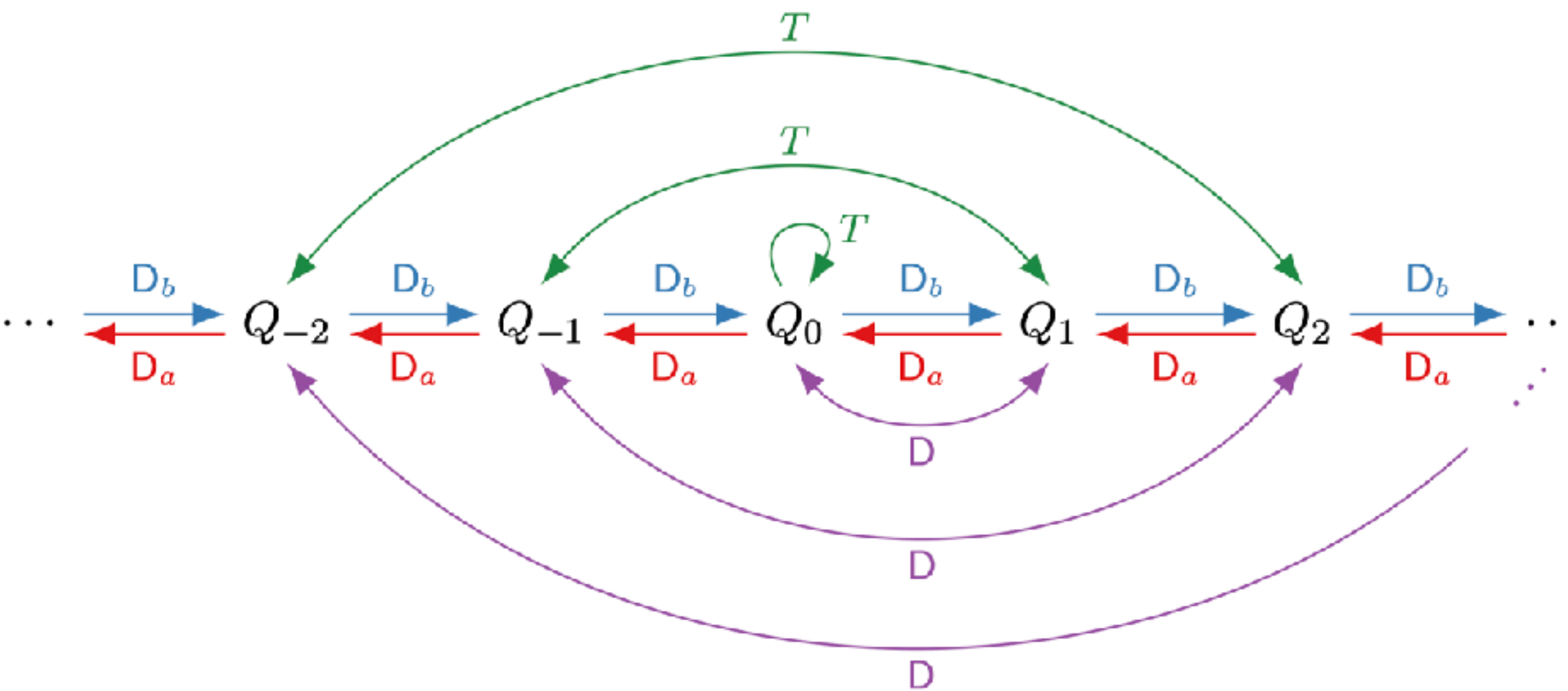


$$= \begin{cases} \dfrac{1}{\sqrt{8}} \begin{pmatrix} \mathbf{1} - Z_j + X_j + \mathrm{i}Y_j & \mathbf{1} + Z_j + X_j - \mathrm{i}Y_j \\ -\mathbf{1} - Z_j + X_j - \mathrm{i}Y_j & \mathbf{1} - Z_j - X_j - \mathrm{i}Y_j \end{pmatrix} & j \text{ odd}, \\ \dfrac{\mathrm{i}}{\sqrt{8}} \begin{pmatrix} \mathbf{1} + Z_j - \mathrm{i}X_j - Y_j & -\mathbf{1} + Z_j - \mathrm{i}X_j + Y_j \\ \mathbf{1} - Z_j - \mathrm{i}X_j + Y_j & \mathbf{1} + Z_j + \mathrm{i}X_j + Y_j \end{pmatrix} & j \text{ even}. \end{cases}$$

A rich algebraic structure

The Onsager charges have a rich interplay with other conserved operators of the XX model [Jones, Prakash, Fendley '24]

- Let $\mathsf{D}_a = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}Q^{\mathrm{M}}}\mathsf{D}$ and $\mathsf{D}_b = \mathrm{e}^{\mathrm{i}\pi Q^{\mathrm{W}}}\mathsf{D}$



Emergence of $\mathrm{TY}(\mathbb{Z}_2, +)$

The XX model has a continuous family of non-invertible symmetries

$$\mathsf{D}_{\phi,\theta} = \mathrm{e}^{\mathrm{i}\phi Q^{\mathrm{M}}} \mathrm{e}^{\mathrm{i}\theta Q^{\mathrm{W}}} \mathsf{D}$$

- $(\mathsf{D}_{\phi,\theta})^2 = (1 + \mathrm{e}^{\mathrm{i}\pi Q^{\mathrm{M}}}) \mathrm{e}^{\mathrm{i}\phi Q^{\mathrm{M}}} \mathrm{e}^{\mathrm{i}(2\phi+\theta) Q^{\mathrm{W}}} \mathrm{e}^{\frac{\mathrm{i}}{2}(\theta-\pi) Q^{\mathrm{M}}} T$

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The $R=\sqrt{2}$ compact boson CFT has an S^1-family of $\mathrm{TY}(\mathbb{Z}_2,+)$ symmetry operators $\mathcal{D}_\varphi$ [Thorngren, Wang '21]

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$$\mathsf{D}_{\phi,\theta} = \mathrm{e}^{\mathrm{i}\phi Q^{\mathrm{M}}}\mathrm{e}^{\mathrm{i}\theta Q^{\mathrm{W}}}\mathsf{D}$$

- $(\mathsf{D}_{\phi,\theta})^2 = (1+\mathrm{e}^{\mathrm{i}\pi Q^{\mathrm{M}}})\mathrm{e}^{\mathrm{i}\phi Q^{\mathrm{M}}}\mathrm{e}^{\mathrm{i}(2\phi+\theta)Q^{\mathrm{W}}}\mathrm{e}^{\frac{\mathrm{i}}{2}(\theta-\pi)Q^{\mathrm{M}}}T$

The $R=\sqrt{2}$ compact boson CFT has an S^1-family of $\mathrm{TY}(\mathbb{Z}_2,+)$ symmetry operators $\mathcal{D}_{\varphi}$ [Thorngren, Wang '21]

In the IR, $T \xrightarrow{\text{IR limit}} \mathrm{e}^{\mathrm{i}\pi(\mathcal{Q}^{\mathcal{M}}+\mathcal{Q}^{\mathcal{W}})}$ [Metlitski, Thorngren '17; Cheng, Seiberg '22]

$$\mathsf{D}_{\phi,\pi-2\phi} \xrightarrow{\text{IR limit}} \mathcal{D}_{\phi}$$

Expressions of the Onsager charges 1

- The Onsager algebra. Formed by conserved charges $\{Q_n, G_n\}$

$$[Q_n, Q_m] = \mathrm{i} G_{m-n} \qquad\qquad [G_n, G_m] = 0$$

$$[Q_n, G_m] = 2\mathrm{i}\left(Q_{n-m} - Q_{n+m}\right)$$

The Onsager charges Q_n in terms of Q^{M} and Q^{W} are

$$Q_n = \begin{cases} 2 S_n Q^{\mathrm{W}} S_n^{-1} & n \text{ odd} \\ S_n Q^{\mathrm{M}} S_n^{-1} & n \text{ even} \end{cases}$$

- Where $S_0 = S_1 = 1, \quad S_2 = \mathrm{e}^{\mathrm{i}\pi Q^{\mathrm{W}}}, \quad S_3 = \mathrm{e}^{\mathrm{i}\pi Q^{\mathrm{W}}} \mathrm{e}^{\mathrm{i}\frac{\pi}{2} Q^{\mathrm{M}}}, \quad \cdots$
- S are the pivots of Onsager algebra [Jones, Prakash, Fendley '24]

Expressions of the Onsager charges 2

$$Q_n = \begin{cases} \frac{1}{2}\sum_{j=1}^{L} Z_j & n=0, \\ \frac{(-1)^{\frac{n+2}{2}}}{2}\sum_{j=1}^{L/2}\left(X_{2j-1}\ \prod_{k=2j}^{2j+n-2} Z_k\ X_{2j+n-1} + Y_{2j}\ \prod_{k=2j+1}^{2j+n-1} Z_k\ Y_{2j+n}\right) & n>0 \text{ even}, \\ \frac{(-1)^{\frac{n-1}{2}}}{2}\sum_{j=1}^{L/2}\left(X_{2j-1}\ \prod_{k=2j}^{2j+n-2} Z_k\ Y_{2j+n-1} - Y_{2j}\ \prod_{k=2j+1}^{2j+n-1} Z_k\ X_{2j+n}\right) & n>0 \text{ odd}, \\ \frac{(-1)^{\frac{n-2}{2}}}{2}\sum_{j=1}^{L/2}\left(Y_{2j+n-1}\ \prod_{k=2j+n}^{2j-2} Z_k\ Y_{2j-1} + X_{2j+n}\ \prod_{k=2j+n+1}^{2j-1} Z_k\ X_{2j}\right) & n<0 \text{ even}, \\ \frac{(-1)^{\frac{n+1}{2}}}{2}\sum_{j=1}^{L/2}\left(X_{2j+n-1}\ \prod_{k=2j+n}^{2j-2} Z_k\ Y_{2j-1} - Y_{2j+n}\ \prod_{k=2j+n+1}^{2j-1} Z_k\ X_{2j}\right) & n<0 \text{ odd}, \end{cases}$$

$$G_n = \begin{cases} \operatorname{sign}(n)\frac{(-1)^{\frac{n}{2}}}{2}\sum_{j=1}^{L/2}(-1)^j\left(X_jY_{j+n}+Y_jX_{j+n}\right)\prod_{k=j+1}^{j+n-1} Z_k & n \text{ even}, \\ \operatorname{sign}(n)\frac{(-1)^{\frac{n-1}{2}}}{2}\sum_{j=1}^{L/2}(-1)^j\left(X_jX_{j+n}-Y_jY_{j+n}\right)\prod_{k=j+1}^{j+n-1} Z_k & n \text{ odd}. \end{cases}$$

Fermionizing by gauging

Gauss law

$$G_j = (-1)^j Z_j \, \mathrm{i} a_{j,j+1} b_{j,j+1}$$

Unitary transformation

$$Z_j \to Z_j \, \mathrm{i} a_{j,j+1} b_{j,j+1} \,, \qquad X_j \to \begin{cases} X_j & j \text{ odd} \,, \\ X_j \, \mathrm{i} a_{j,j+1} b_{j,j+1} & j \text{ even} \,. \end{cases}$$

$$a_{j,j+1} \to \begin{cases} X_j \, a_{j,j+1} & j \text{ odd} \,, \\ Y_j \, a_{j,j+1} & j \text{ even} \,, \end{cases} \qquad b_{j,j+1} \to \begin{cases} -X_j \, b_{j,j+1} & j \text{ odd} \,, \\ Y_j \, b_{j,j+1} & j \text{ even} \,. \end{cases}$$

Qubits now polarized $Z_j = 1$

Bosonizing by gauging

Gauss law

$$G_j = \begin{cases} X_{j-1,j}\,(\mathrm{i}a_j b_{j+1})\,Y_{j,j+1} & j \text{ odd},\\ -Y_{j-1,j}\,(\mathrm{i}a_j b_{j+1})\,X_{j,j+1} & j \text{ even}. \end{cases}$$

Unitary transformation

$$a_j \to \begin{cases} -X_{j-1,j}\,a_j & j \text{ odd},\\ Y_{j-1,j}\,a_j & j \text{ even}, \end{cases} \qquad b_j \to \begin{cases} -X_{j-1,j}\,b_j & j \text{ odd},\\ -Y_{j-1,j},\,b_j & j \text{ even}, \end{cases}$$

$$X_{j-1,j} \to \begin{cases} X_{j-1,j} & j \text{ odd},\\ X_{j-1,j}\,(\mathrm{i}a_j\,b_j) & j \text{ even}, \end{cases} \qquad Z_{j-1,j} \to (-1)^{j-1}\,Z_{j-1,j}\,(\mathrm{i}a_j\,b_j).$$

Fermions now polarized $\mathrm{i}a_j b_j = 1$