Infinite-Order Lattice Anomalies and CPT

Sal Pace

IAS Princeton



CCNY Seminar



Shu-Heng
Shao



Elijah
Lew-Smith

arXiv:2606.12510

Quantum Systems

Two primary frameworks for describing quantum systems:

Quantum lattice models (QLMs)*

Quantum field theories (QFTs)

- Incredibly wide-reaching: cond-mat, hep, quant-ph, & math.

*Discrete space, continuous time

Quantum Systems

Two primary frameworks for describing quantum systems:

Quantum lattice models (QLMs)*

Quantum field theories (QFTs)

- Incredibly wide-reaching: cond-mat, hep, quant-ph, & math.

Different frameworks, but intimately related:

$$\{\text{QLMs}\} \xrightarrow{\text{continuum limit}} \{\text{QFTs}\}$$

- It's important to better understand this relation.

*Discrete space, continuous time

Kinematics

A first step is to work at the level of kinematics:

$$\{\text{QLM symmetries}\} \longrightarrow \{\text{QFT symmetries}\}$$

Kinematics

A first step is to work at the level of kinematics:

$$\{\text{QLM symmetries}\} \longrightarrow \{\text{QFT symmetries}\}$$

- Some interesting questions in this arena are:

1. What is {symmetries} for QLMs and QFTs?
2. Given a QLM symmetry, which QFT symmetries can it become?
3. Given a QFT symmetry, which QLM symmetries can it arise from?

Kinematics

A first step is to work at the level of kinematics:

$$\{\text{QLM symmetries}\} \longrightarrow \{\text{QFT symmetries}\}$$

- Some interesting questions in this arena are:

  1. What is {symmetries} for QLMs and QFTs?
  2. Given a QLM symmetry, which QFT symmetries can it become?
  3. Given a QFT symmetry, which QLM symmetries can it arise from?

Theme of this talk: question 3 for invertible 0-form symmetries.

- Which QFT anomalies can arise from QLM anomalies?
- QLMs with Hilbert space $\bigotimes_i \mathscr{H}_i$ and finite $\dim \mathscr{H}_i$

Kinematics

Why care?

Practical QFT reason: A better "lattice laboratory" for QFTs

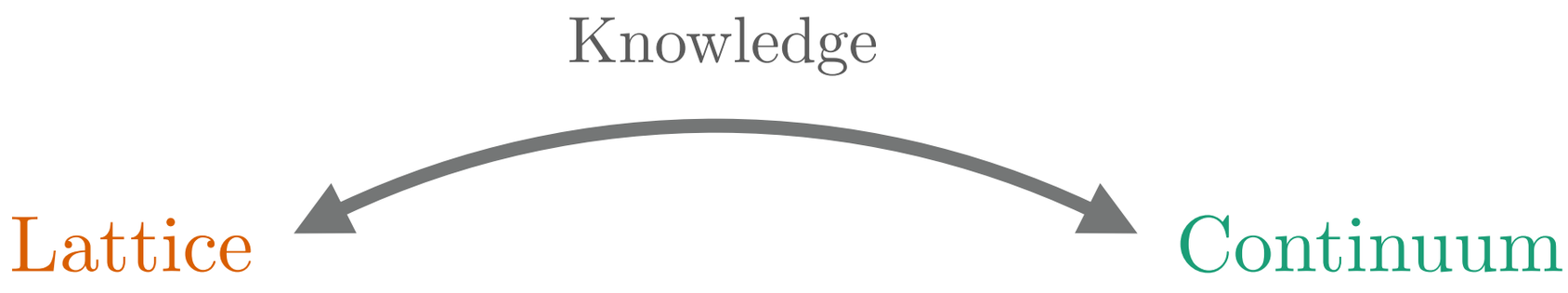
Practical QLM reason: Leads to new anomalies in QLMs
$\Longrightarrow$ new constraints on phases of quantum matter

Conceptual reason: The interplay between QLMs and QFTs continually pushes each other forward.

Theme of this talk: question 3 for invertible 0-form symmetries.

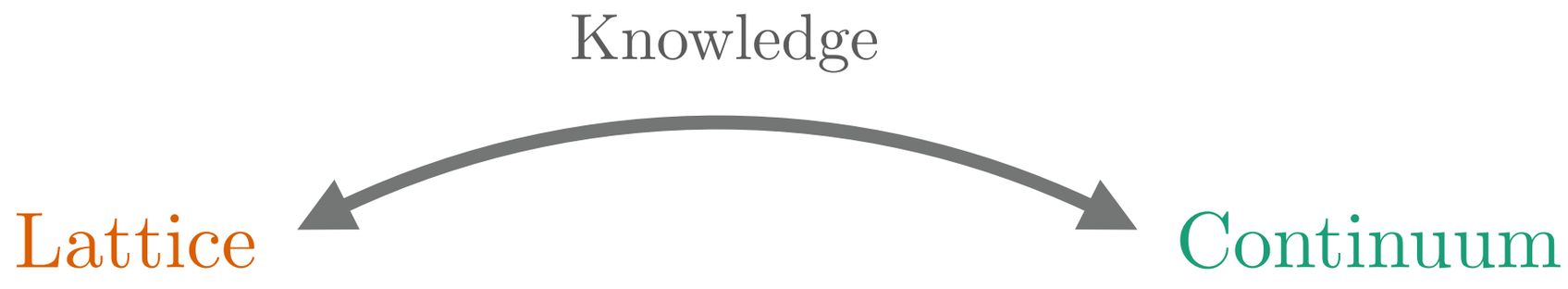
- Which QFT anomalies can arise from QLM anomalies?
- QLMs with Hilbert space $\bigotimes_i \mathscr{H}_i$ and finite dim$\mathscr{H}_i$

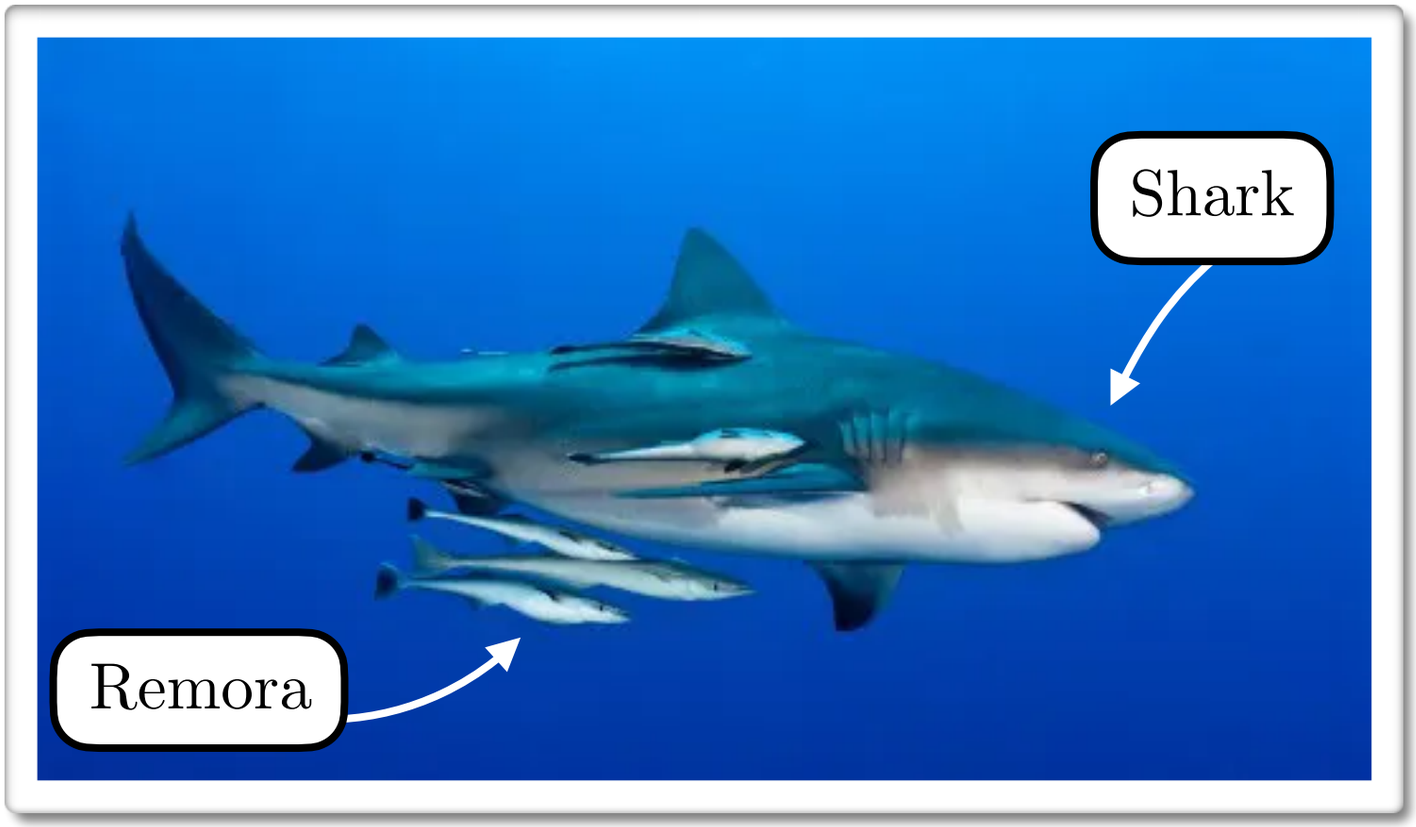
Kinematics



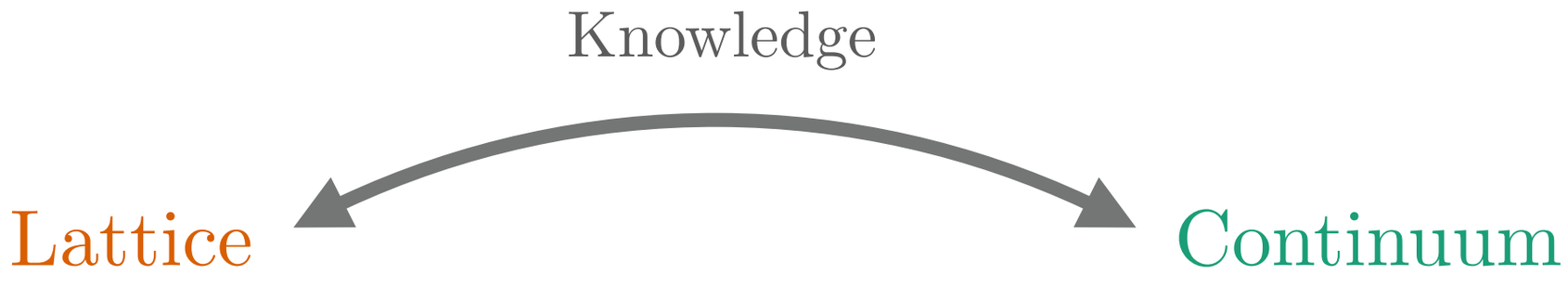
Mutualistic relationship

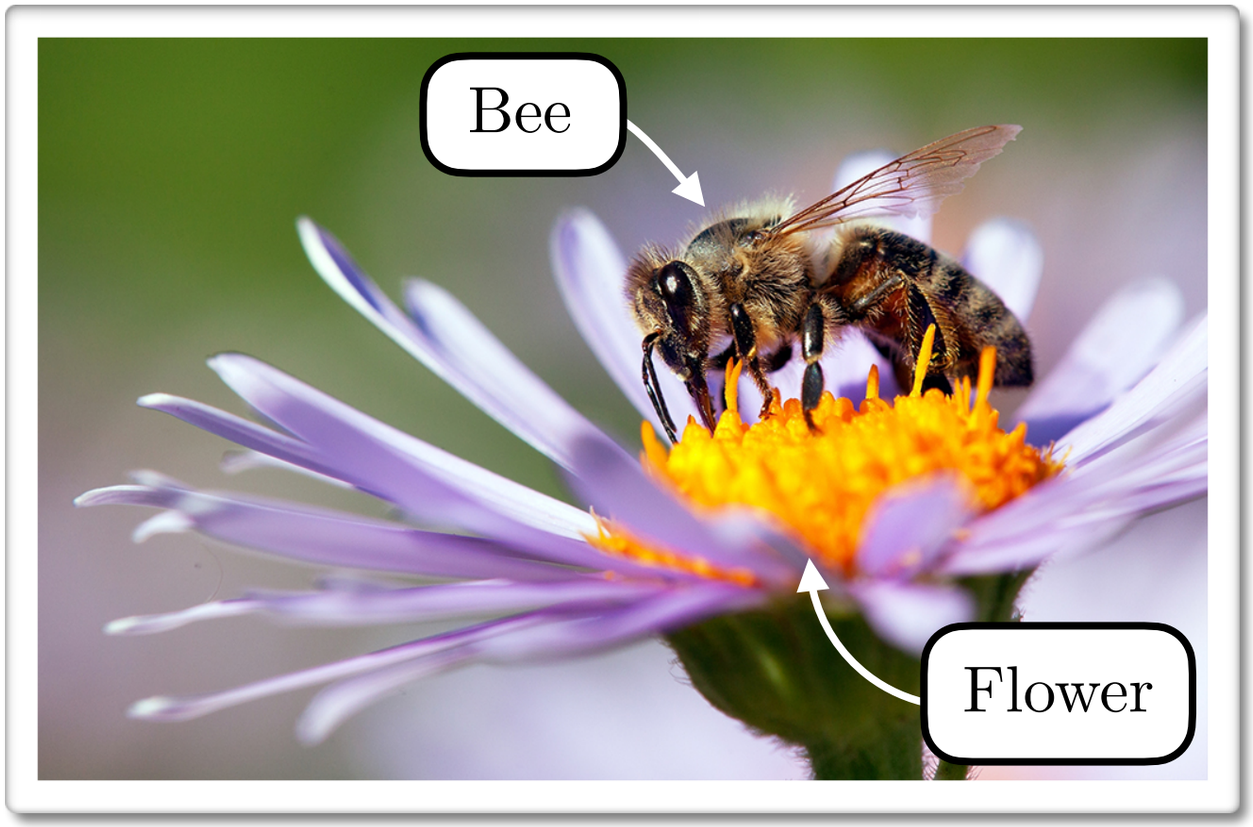
Kinematics





Kinematics





Kinematics

Knowledge

Outline for this talk

1) A light review of anomalies of symmetries

2) The Onsager symmetry: Lattice realization of a 1+1d Dirac fermion's vector and axial symmetries
[*Chatterjee, **SP**, Shao '24*]

3) Anomalies from Onsager + lattice CPT symmetry
[*Lew-Smith, **SP**, Shao '26*]



Anomalies

't Hooft anomalies = obstruction* to gauging a global symmetry

$$Z[A + \mathrm{d}f] = Z[A]\, \mathrm{e}^{\mathrm{i}\theta(A,f)}$$

- Constrains IR dynamics: IR theory $\neq$ trivial theory.

  IR must be gapless, have SSB or topological order, *etc*

- Not a binary "yes or no" thing: The same symmetry group can have many inequivalent 't Hooft anomalies

*Modulo local counterterms

Anomalies

't Hooft anomalies = obstruction* to gauging a global symmetry

$$Z[A + \mathrm{d}f] = Z[A]\, \mathrm{e}^{\mathrm{i}\theta(A,f)}$$

- Constrains IR dynamics: IR theory $\neq$ trivial theory.

IR must be gapless, have SSB or topological order, *etc*

- Not a binary "yes or no" thing: The same symmetry group can have many inequivalent 't Hooft anomalies

't Hooft anomaly $\neq$ a "symmetry of the classical theory fails to be a symmetry of the quantum theory" type of anomaly

- Related to 't Hooft anomalies by gauging
- This talk: Anomalies in QFT $\equiv$ 't Hooft anomalies in QFT

*Modulo local counterterms

Anomalies

't Hooft anomalies = obstruction* to gauging a global symmetry

$$Z[A + \mathrm{d}f] = Z[A]\, \mathrm{e}^{\mathrm{i}\theta(A,f)}$$

- Constrains IR dynamics: IR theory $\neq$ trivial theory
- 't
- b
- Related to 't Hooft anomalies by gauging
- This talk: Anomalies in QFT $\equiv$ 't Hooft anomalies in QFT

The theory of anomalies of ordinary symmetries in relativistic QFT is by now a mature subject.

Understanding anomalies in QLMs is an increasingly active direction!

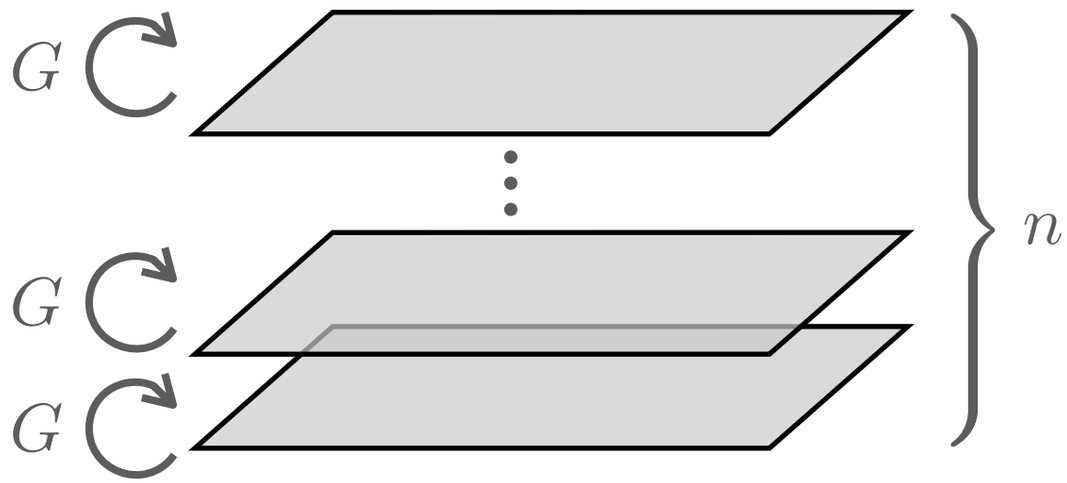
[*Bols, Carvalho, Chatterjee, Chen, Cheng, Czajka, De Roeck, De Wilde, Else, Feng, Geiko, Hsin, Ji, Kapustin, Kawagoe, Kim, Kobayashi, Levin, Liu, Long,* ***SP****, Ryu, Seiberg, Seifnashri, Shao, Shirley, Sopenko, Spodyneiko, Thorngren, Tu, Xu, Yi, Zhang, Zou,* ⋯]

*Modulo local counterterms

The Order of an Anomaly

A key characterization of an anomaly ω is its order $\mathrm{ord}(\omega)$

- Smallest positive integer n such that the diagonal G symmetry of the n-copy system is anomaly-free.

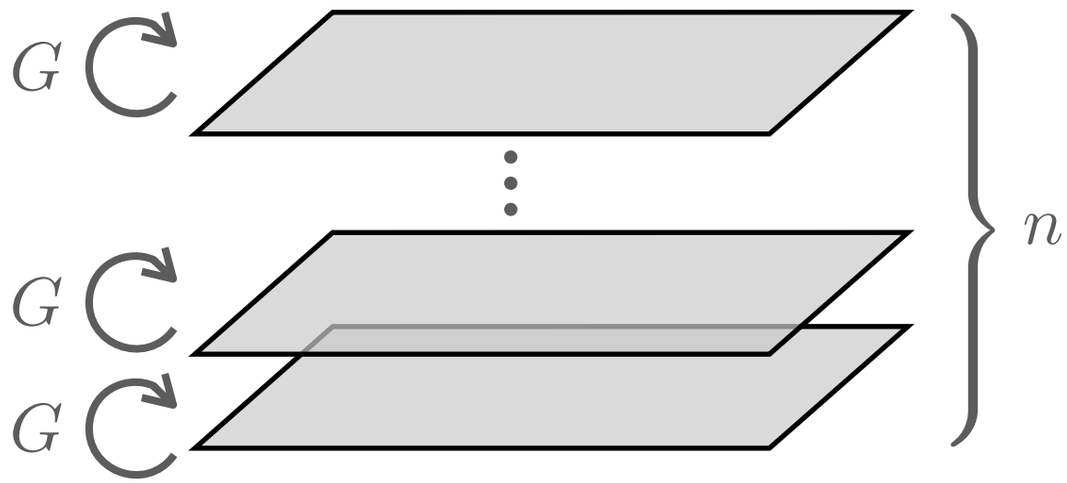


- $\mathrm{ord}(\omega) = 1$ if and only if the G symmetry is anomaly free.

The Order of an Anomaly

A key characterization of an anomaly ω is its order $\mathrm{ord}(\omega)$

- Smallest positive integer n such that the diagonal G symmetry of the n-copy system is anomaly-free.



- $\mathrm{ord}(\omega) = 1$ if and only if the G symmetry is anomaly free.

If $\mathrm{ord}(\omega)$ exists $\Longrightarrow$ ω is a finite-order anomaly.

- Otherwise, ω is an infinite-order anomaly: $\mathrm{ord}(\omega) = \infty$

From the UV to the IR and Back

Given a UV model with G_{UV} symmetry and anomaly ω_{UV} :

Taking the IR limit induces a homomorphism

$$\rho \colon G_{\mathrm{UV}} \to G_{\mathrm{IR}}$$

From the UV to the IR and Back

Given a UV model with G_{UV} symmetry and anomaly ω_{UV}:

Taking the IR limit induces a homomorphism

$$\rho\colon G_{\mathrm{UV}} \to G_{\mathrm{IR}}$$

- In QFT, with $\mathrm{Anom}(G)$ the group of inequivalent G anomalies

$$\rho^*\colon \mathrm{Anom}(G_{\mathrm{IR}}) \to \mathrm{Anom}(G_{\mathrm{UV}})$$

From the UV to the IR and Back

Given a UV model with G_{UV} symmetry and anomaly ω_{UV}:

Taking the IR limit induces a homomorphism

$$\rho\colon G_{\mathrm{UV}} \to G_{\mathrm{IR}}$$

- In QFT, with $\mathrm{Anom}(G)$ the group of inequivalent G anomalies

$$\rho^*\colon \mathrm{Anom}(G_{\mathrm{IR}}) \to \mathrm{Anom}(G_{\mathrm{UV}})$$

Anomaly matching: $\rho^*(\omega_{\mathrm{IR}}) = \omega_{\mathrm{UV}}$

- When the UV and IR symmetries are the same:

$$G_{\mathrm{UV}} \cong G_{\mathrm{IR}} \;\&\; \rho = \mathrm{id} \implies \omega_{\mathrm{IR}} = \omega_{\mathrm{UV}}$$

- A nontrivial UV anomaly implies a nontrivial IR anomaly:

$$\omega_{\mathrm{UV}} = \rho^*(\omega_{\mathrm{IR}}) \neq 1 \implies \omega_{\mathrm{IR}} \neq 1$$

From the UV to the IR and Back

An anomaly's order can change under the IR limit

- The anomaly matching condition implies that

$$\mathrm{ord}(\omega_{\mathrm{UV}}) \leq \mathrm{ord}(\omega_{\mathrm{IR}})$$

$$\omega_{\mathrm{UV}} = \rho^{*}(\omega_{\mathrm{IR}}) \neq 1 \implies \omega_{\mathrm{IR}} \neq 1$$

From the UV to the IR and Back

An anomaly's order can change under the IR limit

- The anomaly matching condition implies that

$$\mathrm{ord}(\omega_{\mathrm{UV}}) \leq \mathrm{ord}(\omega_{\mathrm{IR}})$$

- Proof for finite-order IR anomalies:

$$\begin{aligned}
&\rho^*(\omega_{\mathrm{IR}}) = \omega_{\mathrm{UV}} \\
\implies\ &\rho^*(\omega_{\mathrm{IR}})^{\mathrm{ord}(\omega_{\mathrm{IR}})} = \omega_{\mathrm{UV}}^{\mathrm{ord}(\omega_{\mathrm{IR}})} \\
\implies\ &1 = \omega_{\mathrm{UV}}^{\mathrm{ord}(\omega_{\mathrm{IR}})} \\
\implies\ &\mathrm{ord}(\omega_{\mathrm{IR}}) = k\,\mathrm{ord}(\omega_{\mathrm{UV}}) \text{ for some } k \in \mathbb{Z}_{>0} \\
\implies\ &\mathrm{ord}(\omega_{\mathrm{UV}}) \leq \mathrm{ord}(\omega_{\mathrm{IR}})
\end{aligned}$$

$$\omega_{\mathrm{UV}} = \rho^*(\omega_{\mathrm{IR}}) \neq 1 \implies \omega_{\mathrm{IR}} \neq 1$$

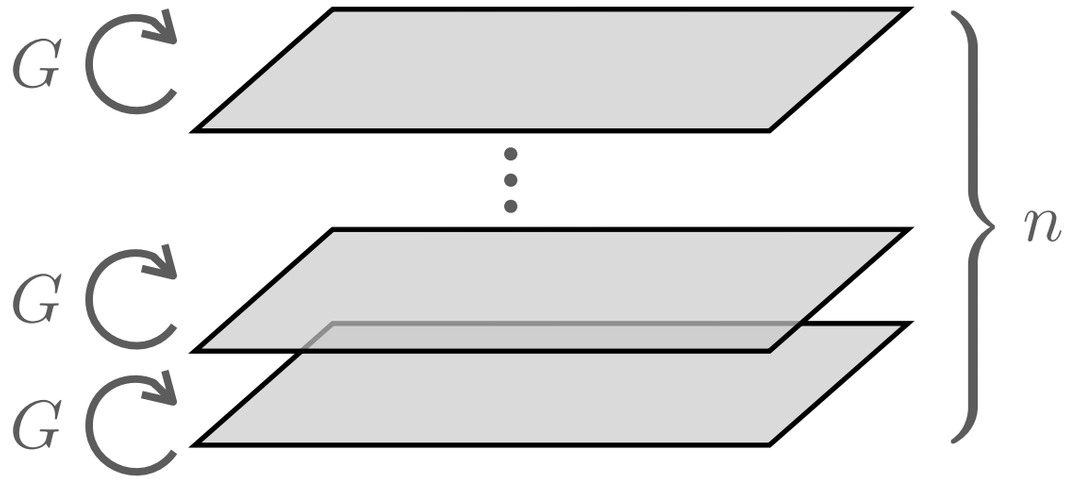
Recap

We just introduced three ideas:

1. An anomaly in QFT:

$$Z[A + \mathrm{d}f] = Z[A]\,\mathrm{e}^{\mathrm{i}\theta(A,f)}$$

2. The order of an anomaly:



3. Changes from the UV to the IR:

$$\mathrm{ord}(\omega_{\mathrm{UV}}) \leq \mathrm{ord}(\omega_{\mathrm{IR}})$$

Quantum Mechanics Example

$0+1$d example: two qubits with Hamiltonian

$$H = (3 - X_1)(1 + Z_2), \qquad \text{Energy spectrum} = \{0, 0, 4, 8\}$$

Quantum Mechanics Example

$0+1$d example: two qubits with Hamiltonian

$$H = (3 - X_1)(1 + Z_2), \qquad \text{Energy spectrum} = \{0, 0, 4, 8\}$$

$G_{\text{UV}} = D_8$ generated by $A = \text{CZ}_{1,2}$, $B = X_1$, and $C = Z_2$:

$$A^2 = B^2 = C^2 = 1, \qquad AB = CBA.$$

- Anomaly-free: $\text{ord}(\omega_{\text{UV}}) = 1$

Quantum Mechanics Example

$0+1$d example: two qubits with Hamiltonian

$$H = (3 - X_1)(1 + Z_2), \qquad \text{Energy spectrum} = \{0, 0, 4, 8\}$$

$G_{\text{UV}} = D_8$ generated by $A = \text{CZ}_{1,2}$, $B = X_1$, and $C = Z_2$:

$$A^2 = B^2 = C^2 = 1, \qquad AB = CBA.$$

- Anomaly-free: $\text{ord}(\omega_{\text{UV}}) = 1$

The IR (ground state space) is the $Z_2 = -1$ eigenspace

- UV D_8 symmetry becomes $\mathbb{Z}_2 \times \mathbb{Z}_2$

$$A_{\text{IR}}^2 = B_{\text{IR}}^2 = 1, \qquad A_{\text{IR}} B_{\text{IR}} = -B_{\text{IR}} A_{\text{IR}}$$

- Nontrivial anomaly: $\text{ord}(\omega_{\text{IR}}) = 2$

Anomaly inequality: $\text{ord}(\omega_{\text{UV}}) = 1 \leq \text{ord}(\omega_{\text{IR}}) = 2$

tl;dr

Infinite-order anomalies are elusive in QLMs.*

- No-go theorems for compact locality-preserving symmetries

  [*Kapustin, Sopenko '24; Liu '26*]

- In contrast, infinite-order anomalies are ubiquitous in QFT (e.g., perturbative anomalies)

*Of the type we discuss here with locally finite-dimensional, onsite tensor-product Hilbert spaces.

tl;dr

Infinite-order anomalies are elusive in QLMs.*

- No-go theorems for compact locality-preserving symmetries
  [*Kapustin, Sopenko '24; Liu '26*]
- In contrast, infinite-order anomalies are ubiquitous in QFT (e.g., perturbative anomalies)

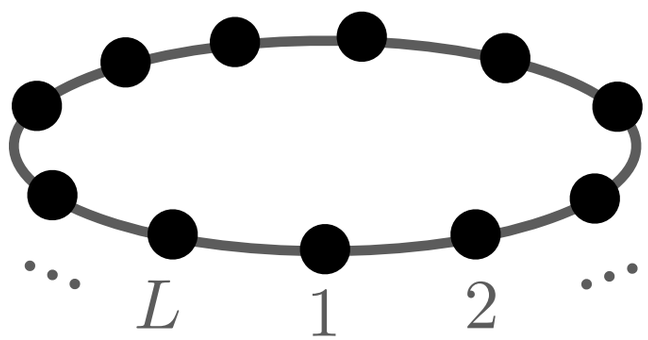
Here, we consider $1+1$d fermionic QLMs and show [*Lew-Smith, **SP**, Shao '26*]

QLM	UV Symmetry	IR Symmetry	Anomaly Order UV $\to$ IR
Staggered Fermion	Onsager [*Chatterjee, **SP**, Shao '24*]	$\frac{\mathrm{U}(1)_\mathrm{V} \times \mathrm{U}(1)_\mathrm{A}}{\mathbb{Z}_2}$	$2 \to \infty$
	Onsager $\rtimes \mathbb{Z}_2^{\mathrm{CPT}}$	$\frac{\mathrm{U}(1)_\mathrm{V} \times \mathrm{U}(1)_\mathrm{A}}{\mathbb{Z}_2} \times \mathbb{Z}_2^{\mathrm{CPT}}$	$\infty \to \infty$

*Of the type we discuss here with locally finite-dimensional, onsite tensor-product Hilbert spaces.

A Simple Lattice Model

Consider a finite lattice on a ring with even L sites



Fermion operator c_j at each site j

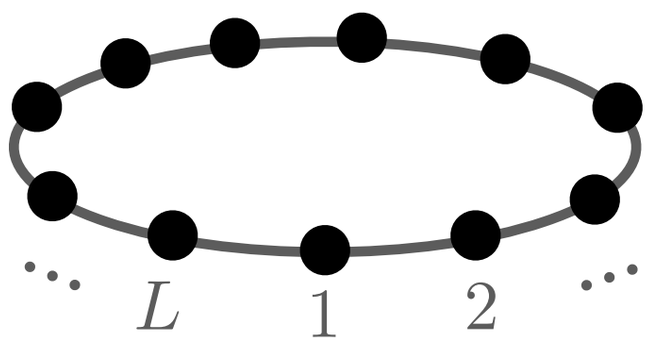
$$\{c_j, c_{j'}^\dagger\} = \delta_{j,j'} \qquad \{c_j, c_{j'}\} = 0$$

Staggered fermion Hamiltonian $H = \mathrm{i} \sum_{j=1}^{L} \left(c_j^\dagger c_{j+1} - c_{j+1}^\dagger c_j \right)$

[*Banks, Susskind, Kogut '76*]

A Simple Lattice Model

Consider a finite lattice on a ring with even L sites



Fermion operator c_j at each site j

$$\{c_j, c_{j'}^\dagger\} = \delta_{j,j'} \qquad \{c_j, c_{j'}\} = 0$$

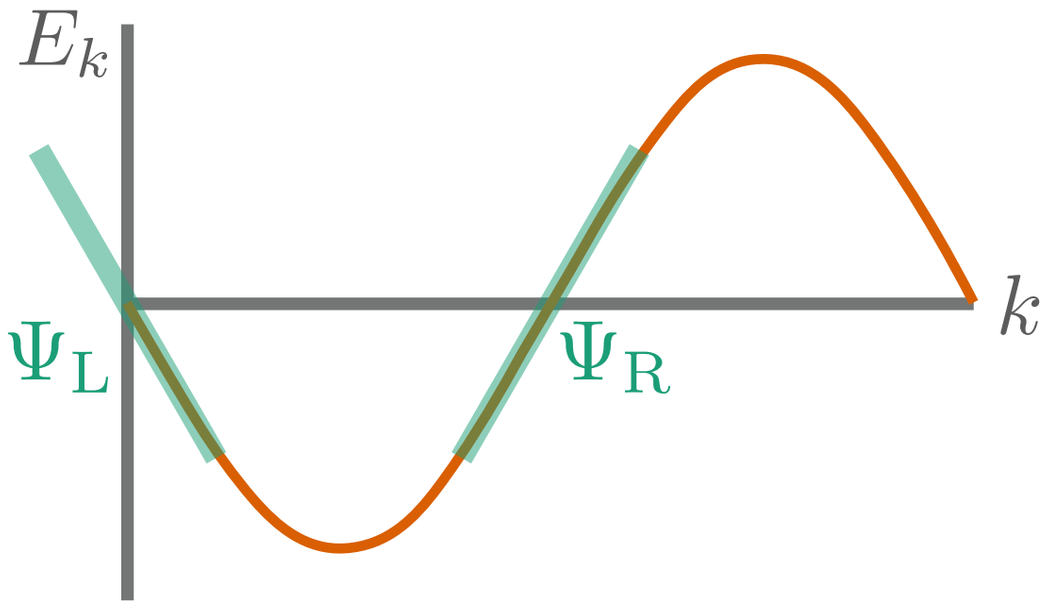
Staggered fermion Hamiltonian $H = \mathrm{i} \sum_{j=1}^{L} \left(c_j^\dagger c_{j+1} - c_{j+1}^\dagger c_j \right)$

[*Banks, Susskind, Kogut '76*]

- H becomes a free, massless Dirac fermion theory in IR
- In momentum space

$$H = \sum_{k \in \mathrm{BZ}} E_k \, c_k^\dagger c_k$$

$$E_k = -2 \sin(k)$$



Onsager Symmetry I: What is it

The symmetry of 1+1d massless Dirac fermion $\Psi = (\Psi_{\mathrm{L}} \quad \Psi_{\mathrm{R}})^{\mathrm{T}}$

$$\mathrm{U}(1)_{\mathrm{V}} : \Psi \mapsto \mathrm{e}^{-\mathrm{i}\theta_{\mathrm{V}}}\Psi \qquad \mathrm{U}(1)_{\mathrm{A}} : \Psi \mapsto \mathrm{e}^{-\mathrm{i}\theta_{\mathrm{A}}\sigma^z}\Psi$$

Does this IR symmetry arise from a UV symmetry of H? Yes!

[*Chatterjee, **SP**, Shao '24*]

Onsager Symmetry I: What is it

The symmetry of 1+1d massless Dirac fermion $\Psi = (\Psi_{\mathrm{L}} \quad \Psi_{\mathrm{R}})^{\mathrm{T}}$

$$\mathrm{U}(1)_{\mathrm{V}} : \Psi \mapsto \mathrm{e}^{-\mathrm{i}\theta_{\mathrm{V}}}\Psi \qquad \mathrm{U}(1)_{\mathrm{A}} : \Psi \mapsto \mathrm{e}^{-\mathrm{i}\theta_{\mathrm{A}}\sigma^z}\Psi$$

Does this IR symmetry arise from a UV symmetry of H? Yes!

[*Chatterjee, **SP**, Shao '24*]

L

- Lattice vector charge $Q_0 = \sum_{j=1} c_j^\dagger c_j$

$$\mathrm{e}^{\mathrm{i}\theta_0 Q_0} : c_k \mapsto \mathrm{e}^{-\mathrm{i}\theta_0} c_k$$

- IR transformation (look at $k = 0$ and $k = \pi$)

$$\Psi_{\mathrm{L}} \mapsto \mathrm{e}^{-\mathrm{i}\theta_0}\Psi_{\mathrm{L}} \qquad \Psi_{\mathrm{R}} \mapsto \mathrm{e}^{-\mathrm{i}\theta_0}\Psi_{\mathrm{R}}$$

Onsager Symmetry I: What is it

The symmetry of 1+1d massless Dirac fermion $\Psi = (\Psi_{\mathrm{L}} \quad \Psi_{\mathrm{R}})^{\mathrm{T}}$

$$\mathrm{U}(1)_{\mathrm{V}} : \Psi \mapsto \mathrm{e}^{-\mathrm{i}\theta_{\mathrm{V}}}\Psi \qquad \mathrm{U}(1)_{\mathrm{A}} : \Psi \mapsto \mathrm{e}^{-\mathrm{i}\theta_{\mathrm{A}}\sigma^z}\Psi$$

Does this IR symmetry arise from a UV symmetry of H? Yes!

[*Chatterjee, **SP**, Shao '24*]

- Lattice axial charge $Q_1 = \dfrac{1}{2}\displaystyle\sum_{j=1}^{L}(c_j^\dagger c_{j+1} + c_j c_{j+1} + \mathrm{h.c.})$

$$\mathrm{e}^{\mathrm{i}\theta_1 Q_1} : c_k \mapsto (\cos\theta_1 - \mathrm{i}\sin\theta_1\cos k)c_k - \sin\theta_1 \sin k\, c^\dagger_{-k}$$

- IR transformation (look at $k = 0$ and $k = \pi$)

$$\Psi_{\mathrm{L}} \mapsto \mathrm{e}^{-\mathrm{i}\theta_1}\Psi_{\mathrm{L}} \qquad \Psi_{\mathrm{R}} \mapsto \mathrm{e}^{\mathrm{i}\theta_1}\Psi_{\mathrm{R}}$$

Onsager Symmetry I: What is it

The symmetry of 1+1d massless Dirac fermion $\Psi = (\Psi_{\mathrm{L}} \quad \Psi_{\mathrm{R}})^{\mathrm{T}}$

$$\mathrm{U}(1)_{\mathrm{V}} \colon \Psi \mapsto \mathrm{e}^{-\mathrm{i}\theta_{\mathrm{V}}}\Psi \qquad \mathrm{U}(1)_{\mathrm{A}} \colon \Psi \mapsto \mathrm{e}^{-\mathrm{i}\theta_{\mathrm{A}}\sigma^z}\Psi$$

Does this IR symmetry arise from a UV symmetry of H? Yes!

[*Chatterjee,* ***SP****, Shao '24*]

Conserved, quantized charges Q_0 and Q_1 have the IR limit

$$Q_0 \xrightarrow{\text{IR}} \mathcal{Q}_{\mathrm{V}} \qquad Q_1 \xrightarrow{\text{IR}} \mathcal{Q}_{\mathrm{A}}$$

- Decomposing in terms of real fermions: $c_j = (a_j + \mathrm{i}b_j)/2$

$$Q_0 = \frac{\mathrm{i}}{2}\sum_{j=1}^{L} a_j b_j \qquad Q_1 = \frac{\mathrm{i}}{2}\sum_{j=1}^{L} a_j b_{j+1}$$

Onsager Symmetry I: What is it

The

Doe

[Chatter

Con

- D

These lattice operators do not commute

$$[Q_0, Q_1] \neq 0 \xrightarrow{\text{IR}} [\mathcal{Q}_V, \mathcal{Q}_A] = 0$$

Q_0 and Q_1 generate the Onsager algebra [*Onsager '44*]

$$Q_n = \frac{\mathrm{i}}{2}\sum_{j=1}^{L} a_j b_{j+n} \qquad G_n = \frac{\mathrm{i}}{2}\sum_{j=1}^{L}(a_j a_{j+n} - b_j b_{j+n})$$

$$[Q_n, Q_m] = \mathrm{i}G_{m-n} \qquad [G_n, G_m] = 0$$

$$[Q_n, G_m] = 2\mathrm{i}(Q_{n-m} - Q_{n+m})$$

UV to IR symmetry map

$$Q_n \xrightarrow{\text{IR}} \begin{cases} \mathcal{Q}_{\text{V}} & n \text{ even} \\ \mathcal{Q}_{\text{A}} & n \text{ odd} \end{cases} \qquad G_n \xrightarrow{\text{IR}} 0$$

Onsager Symmetry I: What is it

The s

Doe

[*Chatter*

Con

➤ D

Remarks on this Onsager symmetry

➤ It is not part of the free-fermion symmetry

$$c_k \mapsto \mathrm{e}^{\mathrm{i}\phi(k)} c_k$$

➤ It is a non-internal symmetry. E.g.,

$$\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}Q_1}\mathrm{e}^{\mathrm{i}\frac{\pi}{2}Q_0} : (a_j, b_j) \mapsto (a_{j-1}, b_{j+1})$$

➤ Its symmetry operators are locality-preserving but generically not quantum cellular automata. E.g.,

$$\mathrm{e}^{\mathrm{i}\lambda(Q_{-1}+Q_1)}$$

➤ Non-compact symmetry group: includes $\mathbb{Z}$ and $\mathbb{R}$ subgroups

Onsager Symmetry II: Its Anomaly

The Dirac fermion's $[\mathrm{U}(1)_\mathrm{V} \times \mathrm{U}(1)_\mathrm{A}]/\mathbb{Z}_2$ symmetry is anomalous

- Manifestation: $\partial_\mu J^\mu_\mathrm{A} = \dfrac{1}{\pi} E_\mathrm{V} \implies$ infinite-order anomaly

Is the Onsager symmetry anomalous?

Onsager Symmetry II: Its Anomaly

The Dirac fermion's $[\mathrm{U}(1)_\mathrm{V} \times \mathrm{U}(1)_\mathrm{A}]/\mathbb{Z}_2$ symmetry is anomalous

➤ Manifestation: $\partial_\mu J^\mu_\mathrm{A} = \dfrac{1}{\pi} E_\mathrm{V} \implies$ infinite-order anomaly

Is the Onsager symmetry anomalous?

What even is an anomaly in a QLM?

Here, we view an anomaly as an obstruction to a trivial* symmetric gapped phase

➤ Avoids having to define gauging

➤ Generalization of the Lieb-Schultz-Mattis theorem

➤ Commonly used for non-invertible symmetry

*Trivial meaning without discrete symmetry-breaking/topological order

Onsager Symmetry II: Its Anomaly

The Dirac fermion's $[\mathrm{U}(1)_\mathrm{V} \times \mathrm{U}(1)_\mathrm{A}]/\mathbb{Z}_2$ symmetry is anomalous

- Manifestation: $\partial_\mu J^\mu_\mathrm{A} = \dfrac{1}{\pi} E_\mathrm{V} \implies$ infinite-order anomaly

Is the Onsager symmetry anomalous?

Onsager Symmetry II: Its Anomaly

The Dirac fermion's $[\mathrm{U}(1)_\mathrm{V} \times \mathrm{U}(1)_\mathrm{A}]/\mathbb{Z}_2$ symmetry is anomalous

- Manifestation: $\partial_\mu J^\mu_\mathrm{A} = \dfrac{1}{\pi} E_\mathrm{V} \implies$ infinite-order anomaly

Is the Onsager symmetry anomalous? Yes! [*Chatterjee, **SP**, Shao '24*]

- Can prove each symmetric local Hamiltonian takes the form

$$H_g = \mathrm{i} \sum_n \sum_{j=1}^{L} g_n \left(a_j a_{j+n} + b_j b_{j+n}\right),$$

  which never has a unique gapped ground state

- Each $H_g \neq 0$ is gapless $\implies$ symmetry-enforced gaplessness

Onsager Symmetry II: Its Anomaly

The Dirac fermion's $[\mathrm{U}(1)_\mathrm{V} \times \mathrm{U}(1)_\mathrm{A}]/\mathbb{Z}_2$ symmetry is anomalous

- Manifestation: $\partial_\mu J^\mu_\mathrm{A} = \dfrac{1}{\pi} E_\mathrm{V} \implies$ infinite-order anomaly

Is the Onsager symmetry anomalous? Yes! [*Chatterjee, **SP**, Shao '24*]

- Can prove each symmetric local Hamiltonian takes the form

$$H_g = \mathrm{i} \sum_n \sum_{j=1}^{L} g_n \left(a_j a_{j+n} + b_j b_{j+n}\right),$$

  which never has a unique gapped ground state

- Each $H_g \neq 0$ is gapless $\implies$ symmetry-enforced gaplessness

In the staggered fermion Hamiltonian, the IR Dirac fermion's anomaly arises from the Onsager symmetry's lattice anomaly.

Onsager Symmetry II: Its Anomaly

The Dirac fermion's $[\mathrm{U}(1)_\mathrm{V} \times \mathrm{U}(1)_\mathrm{A}]/\mathbb{Z}_2$ symmetry is anomalous

➤ Manifestation: $\partial_\mu J^\mu_\mathrm{A} = \frac{1}{\pi} E_\mathrm{V} \implies$ infinite-order anomaly

Is th

➤ C

w

➤ Ex

<u>Remarks</u>

➤ Not all Onsager symmetries are anomalous

➤ For this Onsager symmetry, there are non-quadratic but terribly non-local symmetric Hamiltonians

➤ Not all Onsager symmetries disallow local non-quadratic terms [*Vernier, O'Brien, Fendley '18 ;* ***SP****, Chatterjee, Shao '24*]

In the staggered fermion Hamiltonian, the IR Dirac fermion's anomaly arises from the Onsager symmetry's lattice anomaly.

Onsager Symmetry III: Anomaly Order

What is the order of the Onsager symmetry's anomaly?

Onsager Symmetry III: Anomaly Order

What is the order of the Onsager symmetry's anomaly?

It is order two: The diagonal Onsager symmetry of the 2-copy system is anomaly-free [*Lew-Smith, **SP**, Shao '26*]

Onsager Symmetry III: Anomaly Order

What is the order of the Onsager symmetry's anomaly?

It is order two: The diagonal Onsager symmetry of the 2-copy system is anomaly-free [*Lew-Smith, **SP**, Shao '26*]

- 2-copy system has two fermion operators c_j^{I}, c_j^{II} per site j.
- Diagonal Onsager symmetry generated by

$$Q_0^{\mathrm{diag}} = Q_0^{\mathrm{I}} + Q_0^{\mathrm{II}}, \qquad Q_1^{\mathrm{diag}} = Q_1^{\mathrm{I}} + Q_1^{\mathrm{II}}$$

- A symmetric Hamiltonian with a unique gapped ground state:

$$H_{\mathrm{gap}} = \frac{\mathrm{i}}{2} \sum_{j=1}^{L} \left(a_j^{\mathrm{I}} a_j^{\mathrm{II}} + b_j^{\mathrm{I}} b_j^{\mathrm{II}} \right)$$

$\Longrightarrow$ The $(Q_0^{\mathrm{diag}}, Q_1^{\mathrm{diag}})$ Onsager symmetry is anomaly-free

UV to IR Anomaly Order

QLM	UV Symmetry	IR Symmetry	Anomaly Order UV $\to$ IR
Staggered Fermion	Onsager [*Chatterjee,* ***SP****, Shao '24*]	$\frac{\mathrm{U}(1)_{\mathrm{V}} \times \mathrm{U}(1)_{\mathrm{A}}}{\mathbb{Z}_2}$	$2 \to \infty$

UV to IR Anomaly Order

QLM	UV Symmetry	IR Symmetry	Anomaly Order UV $\to$ IR
Staggered Fermion	Onsager [*Chatterjee, **SP**, Shao '24*]	$\frac{\mathrm{U}(1)_\mathrm{V} \times \mathrm{U}(1)_\mathrm{A}}{\mathbb{Z}_2}$	$2 \to \infty$

Deformed 2-copy staggered fermion Hamiltonian

$$\frac{1}{2}\sum_{j=1}^{L}\left(a_j^{\mathrm{I}}a_{j+1}^{\mathrm{I}} + b_j^{\mathrm{I}}b_{j+1}^{\mathrm{I}} + a_j^{\mathrm{II}}a_{j+1}^{\mathrm{II}} + b_j^{\mathrm{II}}b_{j+1}^{\mathrm{II}}\right) + gH_{\mathrm{gap}}$$

- Single-particle dispersions $E_k^{\pm} = -2\sin(k) \pm g$
- Trivially gapped for $|g| > 2$

IR limit of H_{gap} deformation ($g \ll 1$): chemical potential term

$$i\mu(\Psi_{\mathrm{L}}^{\mathrm{I}})^\dagger\Psi_{\mathrm{L}}^{\mathrm{II}} + i\mu(\Psi_{\mathrm{R}}^{\mathrm{I}})^\dagger\Psi_{\mathrm{R}}^{\mathrm{II}} + \mathrm{h.c.}$$

UV to IR Anomaly Order

QLM	UV Symmetry	IR Symmetry	Anomaly Order UV $\to$ IR
Staggered Fermion	Onsager [*Chatterjee, **SP**, Shao '24*]	$\frac{\mathrm{U}(1)_{\mathrm{V}} \times \mathrm{U}(1)_{\mathrm{A}}}{\mathbb{Z}_2}$	$2 \to \infty$

Defor

Can the order of the Onsager symmetry's anomaly be enhanced to infinite order?

➤ One way to increase an anomaly's order is by introducing additional symmetry

Here, we will add a lattice CPT symmetry

➤ Natural symmetry to add from the QFT perspective

➤ S

IR l

➤ C

Lattice CPT Symmetry

Consider the anti-unitary operator Π satisfying

$$\Pi^2 = 1, \qquad \Pi\, \mathrm{i}\, \Pi^\dagger = -\mathrm{i}, \qquad \Pi\, c_j\, \Pi^\dagger = c^\dagger_{-j+1}$$

- Is a $\mathbb{Z}_2$ anti-unitary reflection operator

Lattice CPT Symmetry

Consider the anti-unitary operator Π satisfying

$$\Pi^2 = 1, \qquad \Pi\, \mathrm{i}\, \Pi^\dagger = -\mathrm{i}, \qquad \Pi\, c_j\, \Pi^\dagger = c^\dagger_{-j+1}$$

- Is a $\mathbb{Z}_2$ anti-unitary reflection operator

It is a symmetry of the staggered fermion Hamiltonian

$$H = \mathrm{i} \sum_{j=1}^{L} \left(c^\dagger_j c_{j+1} - c^\dagger_{j+1} c_j \right)$$

- In the IR, Π becomes the CPT symmetry operator of the Dirac fermion QFT

  $\Longrightarrow$ Π is a lattice $\mathbb{Z}_2^{\mathrm{CPT}}$ symmetry operator

Lattice CPT Symmetry

Consider the anti-unitary operator Π satisfying

- Is a

It is a

- In t

Dira

$\Longrightarrow$

Π acts on the Onsager charge operators as

$$\Pi Q_n \Pi^\dagger = -Q_{-n}, \qquad \Pi G_n \Pi^\dagger = G_n$$

- (Q_0, Q_1, Π) form the symmetry group

$$\text{Onsager} \rtimes \mathbb{Z}_2^{\text{CPT}}$$

Lattice CPT Symmetry

Consider the anti-unitary operator Π satisfying

➤ Is a

It is a

➤ In t
Dira

$\Longrightarrow$

Π acts on the Onsager charge operators as

$$\Pi Q_n \Pi^\dagger = -Q_{-n}, \qquad \Pi G_n \Pi^\dagger = G_n$$

➤ (Q_0, Q_1, Π) form the symmetry group

$$\text{Onsager} \rtimes \mathbb{Z}_2^{\text{CPT}}$$

➤ In the IR of the staggered fermion Hamiltonian

$$\Pi Q_n \Pi^\dagger = -Q_{-n} \xrightarrow{\text{IR}} \begin{cases} \Pi_{\text{IR}} \mathcal{Q}_{\text{V}} \Pi_{\text{IR}}^\dagger = -\mathcal{Q}_{\text{V}} & n \text{ even} \\ \Pi_{\text{IR}} \mathcal{Q}_{\text{A}} \Pi_{\text{IR}}^\dagger = -\mathcal{Q}_{\text{A}} & n \text{ odd} \end{cases}$$

$$\text{Onsager} \rtimes \mathbb{Z}_2^{\text{CPT}} \xrightarrow{\text{IR}} [\text{U}(1)_{\text{V}} \times \text{U}(1)_{\text{A}}]/\mathbb{Z}_2 \times \mathbb{Z}_2^{\text{CPT}}$$

Onsager + CPT: Anomaly Order

The Onsager $\rtimes \mathbb{Z}_2^{\mathrm{CPT}}$ symmetry is anomalous

Onsager + CPT: Anomaly Order

The Onsager $\rtimes \mathbb{Z}_2^{\mathrm{CPT}}$ symmetry is anomalous

It's an infinite-order anomaly: The diagonal symmetry of the N-copy system is anomalous for all N. [*Lew-Smith, **SP**, Shao '26*]

Onsager + CPT: Anomaly Order

The Onsager $\rtimes \mathbb{Z}_2^{\mathrm{CPT}}$ symmetry is anomalous

It's an infinite-order anomaly: The diagonal symmetry of the N-copy system is anomalous for all N. [*Lew-Smith, **SP**, Shao '26*]

- N-copy system has N fermion operators $\{c_j^{(\ell)}\}_{\ell=1}^N$ per site j
- Diagonal Onsager $\rtimes \mathbb{Z}_2^{\mathrm{CPT}}$ symmetry generated by

$$Q_0^{\mathrm{diag}} = \sum_{\ell=1}^N Q_0^{(\ell)}, \qquad Q_1^{\mathrm{diag}} = \sum_{\ell=1}^N Q_1^{(\ell)}, \qquad \Pi^{\mathrm{diag}}$$

- Can prove each symmetric local Hamiltonian takes the form

$$\frac{\mathrm{i}}{2} \sum_n \sum_{I,J=1}^N \sum_{j=1}^L g_n^{IJ} \left(a_j^I a_{j+n}^J + b_j^I b_{j+n}^J \right) \qquad (g_{-n}^{IJ} = -g_n^{IJ})$$

  which never has a unique gapped ground state

Recap

QLM	UV Symmetry	IR Symmetry	Anomaly Order UV $\to$ IR
Staggered Fermion	Onsager	$\frac{\mathrm{U}(1)_\mathrm{V} \times \mathrm{U}(1)_\mathrm{A}}{\mathbb{Z}_2}$	$2 \to \infty$

Recap

QLM	UV Symmetry	IR Symmetry	Anomaly Order UV $\to$ IR
Staggered Fermion	Onsager	$\frac{\mathrm{U}(1)_\mathrm{V} \times \mathrm{U}(1)_\mathrm{A}}{\mathbb{Z}_2}$	$2 \to \infty$
	Onsager $\rtimes \mathbb{Z}_2^{\mathrm{CPT}}$	$\frac{\mathrm{U}(1)_\mathrm{V} \times \mathrm{U}(1)_\mathrm{A}}{\mathbb{Z}_2} \times \mathbb{Z}_2^{\mathrm{CPT}}$	$\infty \to \infty$

Recap

QLM	UV Symmetry	IR Symmetry	Anomaly Order UV $\to$ IR
Staggered Fermion	Onsager	$\frac{\mathrm{U}(1)_\mathrm{V} \times \mathrm{U}(1)_\mathrm{A}}{\mathbb{Z}_2}$	$2 \to \infty$
	Onsager $\rtimes \mathbb{Z}_2^{\mathrm{CPT}}$	$\frac{\mathrm{U}(1)_\mathrm{V} \times \mathrm{U}(1)_\mathrm{A}}{\mathbb{Z}_2} \times \mathbb{Z}_2^{\mathrm{CPT}}$	$\infty \to \infty$
Heisenberg Chain	$\mathbb{Z}^{\mathrm{tran}}$	$\mathbb{Z}_2$	$1 \to 2$ [*Metlitski, Thorngren '17*]
	$\mathbb{Z}^{\mathrm{tran}} \rtimes \mathbb{Z}_2^{\mathrm{CPT}}$	$\mathbb{Z}_2 \times \mathbb{Z}_2^{\mathrm{CPT}}$	$2 \to 2$ [*Seiberg, Shao, Zhang '25*]
Kitaev Chain	$\mathbb{Z}^{\mathrm{maj}}$	$\mathbb{Z}_2$	$2 \to 8$ [*Seiberg, Shao '23*]
	$\mathbb{Z}^{\mathrm{maj}} \rtimes \mathbb{Z}_2^{\mathrm{CPT}}$	$\mathbb{Z}_2 \times \mathbb{Z}_2^{\mathrm{CPT}}$	$8 \to 8$ [*Seiberg, Shao, Zhang '25*]

Future Directions

The role of CPT, and more generally CRT, in QLMs

- Interesting results also noted for invertible phases [*Ogata '22; Sopenko '25*]

Systematic study of the anomaly's stability to ancillas

- Preliminary results: the anomaly is robust to a natural class of non-onsite ancillas [*Lew-Smith, **SP**, Shao '26*]

Analogous story for Onsager symmetries in $> 1+1$d

- The anomalies of such Onsager symmetries can realize parity anomalies [***SP**, Kim, Chatterjee, Shao '25*], Witten's SU(2) anomaly [*Gioia, Thorngren '25*], and symmetry-enforced Fermi surfaces [*Kim, **SP**, Shao '25*].

Backup Slides

Anomalies in quantum mechanics

Consider QM model with Hilbert space $\mathscr{H}$ and Hamiltonian H

- Assume there is a unitary G symmetry: $[U_g, H] = 0$, $g \in G$
- Symmetry has an anomaly if

$$U_g U_h = \mathrm{e}^{\mathrm{i}\theta(g,h)} U_{gh} \qquad \mathrm{e}^{\mathrm{i}\theta(g,h)} \neq \mathrm{e}^{\mathrm{i}(f(g)+f(h)-f(gh))}$$

Anomalies in quantum mechanics

Consider QM model with Hilbert space $\mathscr{H}$ and Hamiltonian H

- Assume there is a unitary G symmetry: $[U_g, H] = 0$, $g \in G$
- Symmetry has an anomaly if

$$U_g U_h = \mathrm{e}^{\mathrm{i}\theta(g,h)} U_{gh} \qquad \mathrm{e}^{\mathrm{i}\theta(g,h)} \neq \mathrm{e}^{\mathrm{i}(f(g)+f(h)-f(gh))}$$

Proof:

1. Assume $|\mathrm{gs}\rangle \in \mathscr{H}$ is a unique gapped ground state of H
2. Therefore, $U_g U_h |\mathrm{gs}\rangle = \mathrm{e}^{\mathrm{i}f(h)} U_g |\mathrm{gs}\rangle = \mathrm{e}^{\mathrm{i}(f(g)+f(h))} |\mathrm{gs}\rangle$
3. However, $U_g U_h |\mathrm{gs}\rangle = \mathrm{e}^{\mathrm{i}\theta(g,h)} U_{gh} |\mathrm{gs}\rangle = \mathrm{e}^{\mathrm{i}(\theta(g,h)+f(gh))} |\mathrm{gs}\rangle$
4. Requires $\mathrm{e}^{\mathrm{i}\theta(g,h)} = \mathrm{e}^{\mathrm{i}(f(g)+f(h)-f(gh))} \Longrightarrow$ contradiction

Anomalies in quantum mechanics

Consider QM model with Hilbert space $\mathscr{H}$ and Hamiltonian H

- Assume there is a unitary G symmetry: $[U_g, H] = 0$, $g \in G$
- Symmetry has an anomaly if

$$U_g U_h = \mathrm{e}^{\mathrm{i}\theta(g,h)} U_{gh} \qquad \mathrm{e}^{\mathrm{i}\theta(g,h)} \neq \mathrm{e}^{\mathrm{i}(f(g)+f(h)-f(gh))}$$

Example:

- A qubit, $\mathscr{H} = \mathrm{span}_{\mathbb{C}}\{|\uparrow\rangle, |\downarrow\rangle\}$, with $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry

$$U_{(n,m)} = X^n Z^m \qquad (n,m) \in \mathbb{Z}_2 \times \mathbb{Z}_2$$

- $U_{(n,m)} U_{(n',m')} = (-1)^{mn'} U_{(n+n',m+m')} \implies$ anomaly

Anomalies in quantum mechanics

Consider QM model with Hilbert space $\mathscr{H}$ and Hamiltonian H

- Assume there is a unitary G symmetry: $[U_g, H] = 0$, $g \in G$
- Symmetry has an anomaly if

$$U_g U_h = \mathrm{e}^{\mathrm{i}\theta(g,h)} U_{gh} \qquad \mathrm{e}^{\mathrm{i}\theta(g,h)} \neq \mathrm{e}^{\mathrm{i}(f(g)+f(h)-f(gh))}$$

Example:

- A qubit, $\mathscr{H} = \mathrm{span}_{\mathbb{C}}\{|\uparrow\rangle, |\downarrow\rangle\}$, with $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry

$$U_{(n,m)} = X^n Z^m \qquad (n,m) \in \mathbb{Z}_2 \times \mathbb{Z}_2$$

- $U_{(n,m)} U_{(n',m')} = (-1)^{mn'} U_{(n+n',m+m')} \implies$ anomaly
- Explicit check: $[U_{(n,m)}, H] = 0 \implies H \propto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$