New anomalies in lattice models of fermions

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CTQM Theory Colloquium

Cultural background

One of the most elementary questions in the quantum physics of many degrees of freedom:

- Given a fixed microscopic set up, which macroscopic phenomena can arise?
- Central to various areas of cond-mat, hep-th, and math-ph

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Prototypical example: quantum phases of matter

Microscopic	$\longrightarrow$	Macroscopic
Spatial dimension		{Quantum phases}
Degrees of freedom		
Symmetries		
$\vdots$		

A first organization of quantum phases

The set {Quantum phases} is generally *very* complicated

- Partition it based on the phases' low-energy spectra

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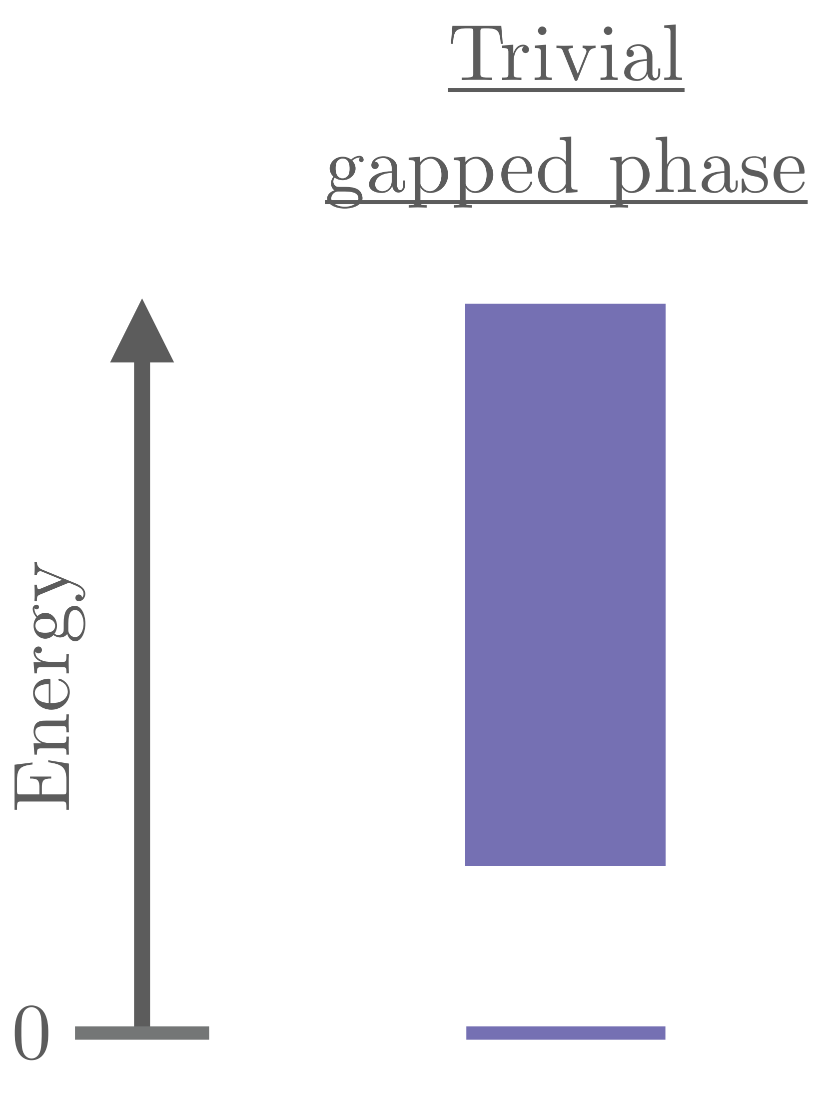

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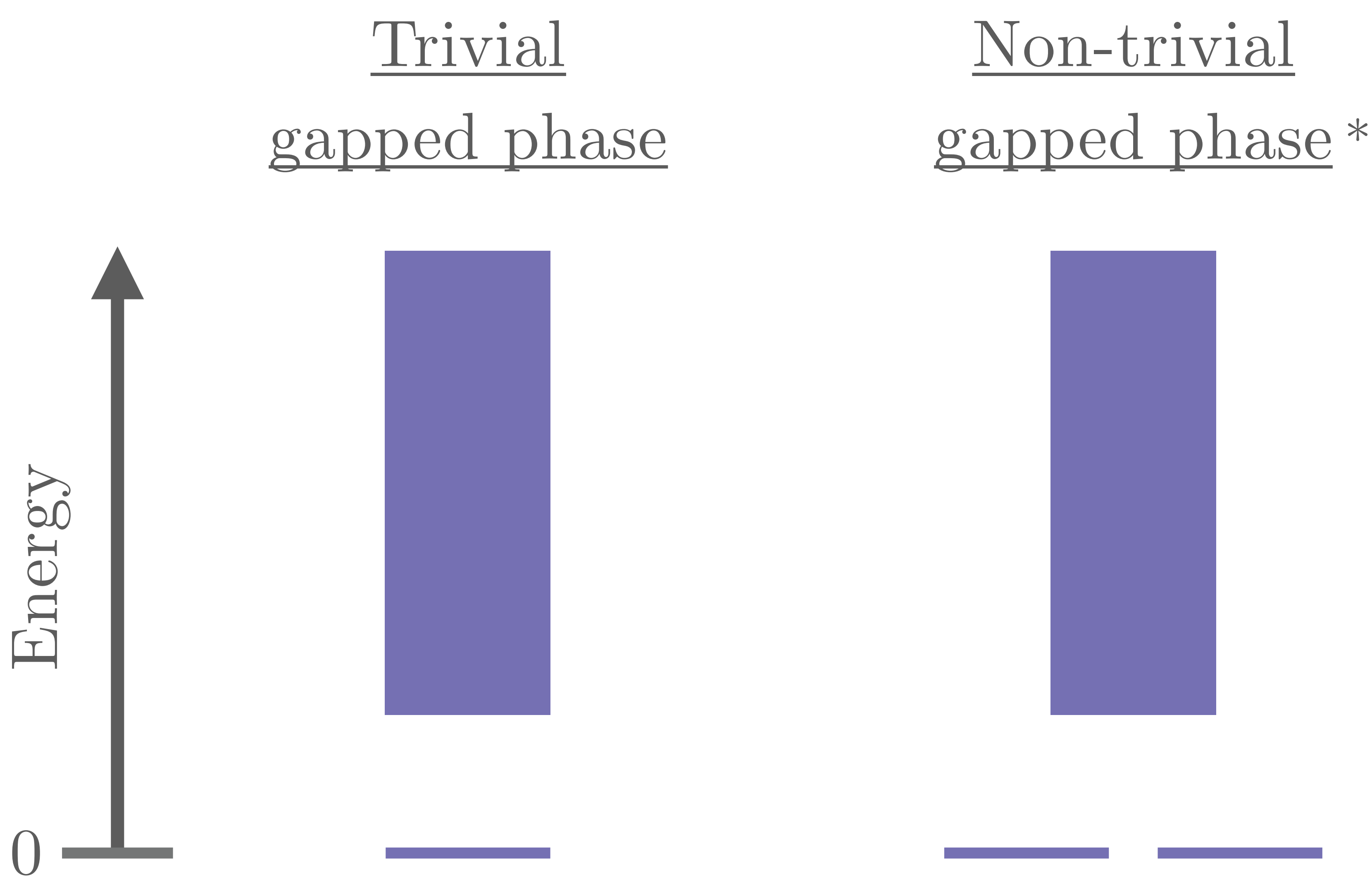

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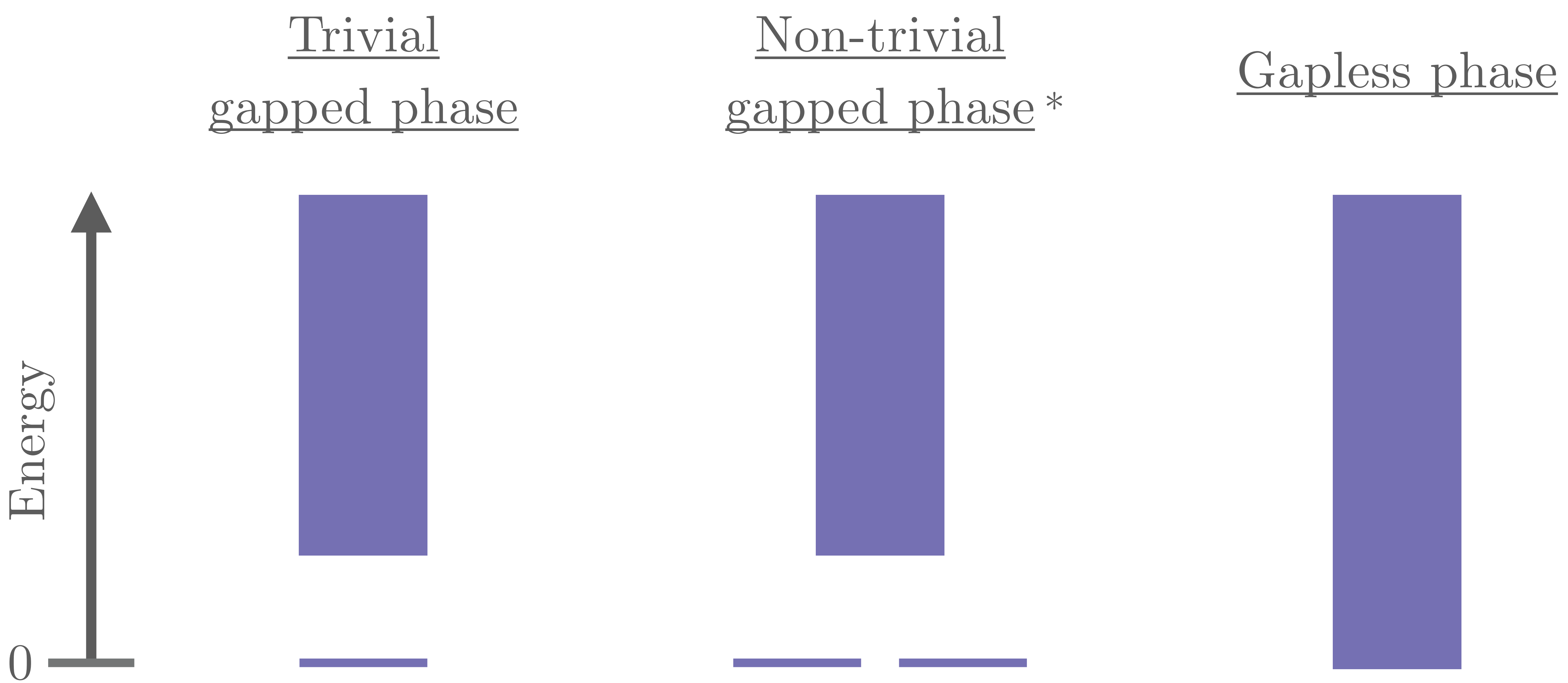

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Gifts from anomalies

Definition: the microscopic data has an anomaly if there exists an obstruction to realizing a trivial gapped phase

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Gifts from anomalies

Definition: the microscopic data has an anomaly if there exists an obstruction to realizing a trivial gapped phase

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Anomalous symmetries are symmetries that are incompatible with a trivial gapped symmetric phase.

- To realize a trivially gapped phase, the anomalous symmetry must be explicitly broken

Gifts from anomalies

Definition: the microscopic data has an anomaly if there

Important disclaimers

1. This definition is different from the "classical symmetry fails to be a quantum symmetry" type anomaly
2. It is a generalization of 't Hooft anomalies in QFT, which are obstructions to gauging a symmetry.
3. It is a popular definition for anomalous spacetime and generalized symmetries

[C.-M. Chang, Y.-H. Lin, S.-H. Shao, Y. Wang, X. Yin '18; X.-G. Wen '18; R. Thorngren, Y. Wang '19; Y. Choi, C. Córdova, P.-S. Hsin, H.T. Lam, S.-H. Shao '21; ⋯ ; W. Shirley, C. Zhang, W. Ji, M. Levin '25]

Anomalies in quantum mechanics

Consider QM model with Hilbert space $\mathscr{H}$ and Hamiltonian H

- Assume there is a unitary G symmetry: $[U_g, H] = 0\,,\ g \in G$
- Symmetry has an anomaly if, for $|\psi\rangle \in \mathscr{H}$,

$$U_g U_h |\psi\rangle = \mathrm{e}^{\mathrm{i}\theta(g,h)} U_{gh} |\psi\rangle \qquad \mathrm{e}^{\mathrm{i}\theta(g,h)} \neq \mathrm{e}^{\mathrm{i}(f(g)+f(h)-f(gh))}$$

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Proof:

1. Assume $|\mathrm{gs}\rangle \in \mathscr{H}$ is a unique gapped ground state of H
2. Therefore, $U_g U_h |\mathrm{gs}\rangle = \mathrm{e}^{\mathrm{i}f(h)} U_g |\mathrm{gs}\rangle = \mathrm{e}^{\mathrm{i}(f(g)+f(h))} |\mathrm{gs}\rangle$
3. However, $U_g U_h |\mathrm{gs}\rangle = \mathrm{e}^{\mathrm{i}\theta(g,h)} U_{gh} |\mathrm{gs}\rangle = \mathrm{e}^{\mathrm{i}(\theta(g,h)+f(gh))} |\mathrm{gs}\rangle$
4. Requires $\mathrm{e}^{\mathrm{i}\theta(g,h)} = \mathrm{e}^{\mathrm{i}(f(g)+f(h)-f(gh))} \implies$ contradiction

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Example:

- A qubit, $\mathscr{H} = \mathrm{span}_{\mathbb{C}}\{|\uparrow\rangle, |\downarrow\rangle\}$, with $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry

$$U_{(n,m)} = X^n Z^m \qquad\qquad (n,m) \in \mathbb{Z}_2 \times \mathbb{Z}_2$$

- $U_{(n,m)} U_{(n',m')} = (-1)^{mn'} U_{(n+n',m+m')} \implies$ anomaly

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Anomalies in >(0+1)D are much richer

- Include the anomalies from QM (i.e., projective representations) and much more due to locality.

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Emergent vs emanant

<u>UV model</u>

symmetries

anomalies

$\xrightarrow[\text{limit}]{\text{Low energy}}$

<u>IR model</u>

symmetries

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The symmetries and anomalies in the IR are generally different from those in the UV

Emergent vs emanant

UV model		IR model
symmetries	$\xrightarrow[\text{limit}]{\text{Low energy}}$	symmetries
anomalies		anomalies

The symmetries and anomalies in the IR are generally different from those in the UV

- For a given UV model, there are two types of IR symmetries and anomalies:
  1) Emergent: have no UV counterpart
  2) Emanant: have a UV counterpart [M. Cheng, N. Seiberg '22]

Emergent vs emanant

What is the definition of emergent?

Emergent (adjective): Arising or coming into being; newly appearing or developing.

What is the definition of emanant?

Emanant (adjective): Flowing out, issuing forth, or radiating from a source.

- For a given UV model, there are two types of IR symmetries and anomalies:

  1) Emergent: have no UV counterpart

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Emergent vs emanant

Five possibilities

Emergent symmetry with no anomaly

Emergent symmetry with emergent anomaly

Emanant symmetry with no anomaly

Emanant symmetry with emanant anomaly

Emanant symmetry with emergent anomaly

1) Emergent: have no UV counterpart

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Lattice vs QFT anomalies

Which QFT anomalies can be realized in lattice models?

- More precisely, can emanate from lattice anomalies

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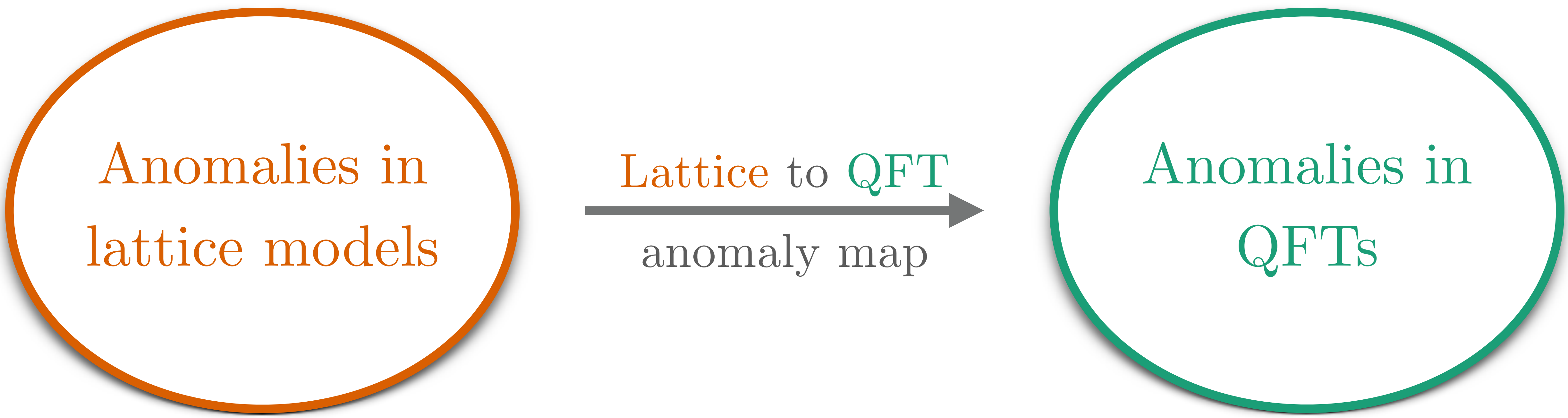

- Surjectivity is not obvious, many elusive QFT anomalies

Lattice vs QFT anomalies

Which QFT anomalies can be realized in lattice models?

- More precisely, can emanate from lattice anomalies

<u>Why care?</u>

<u>Practical reason</u>: a better "lattice laboratory" for QFTs to do numerics and have an intrinsic UV cutoff

<u>Conceptual reason</u>: the interplay between lattice models and QFTs continually push each other forward.

Knowledge

Lattice $\xleftrightarrow{\hspace{3em}}$ Continuum

New anomalies in lattice models of fermions

1) Lattice chiral anomaly: gateway to Onsager symmetries

[Arkya Chatterjee, **Sal Pace**, Shu-Heng Shao, PRL '25 (arXiv:2409.12220)]

2) Lattice parity anomaly: symmetry-enforced Dirac cones

[**Sal Pace**, Luke Kim, Arkya Chatterjee, Shu-Heng Shao, arXiv:2505.04684]

3) Lattice LU(1) anomaly: symmetry-enforced Fermi surfaces

[Luke Kim, **Sal Pace**, Shu-Heng Shao, *to appear*]

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MIT → Stony Brook

Luke Kim
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Shu-Heng Shao
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Dirac fermion field theory

Free, massless Dirac fermion $\Psi = (\Psi_{\mathrm{L}}, \Psi_{\mathrm{R}})^{\mathsf{T}}$ in $1+1$D:

$$\mathscr{L} = \mathrm{i}\,\Psi_{\mathrm{L}}^{\dagger}(\partial_t + \partial_x)\Psi_{\mathrm{L}} + \mathrm{i}\,\Psi_{\mathrm{R}}^{\dagger}(\partial_t - \partial_x)\Psi_{\mathrm{R}}$$

- Ψ_{L} (Ψ_{R}) is a left (right) moving complex Weyl fermion

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Chiral $(\mathrm{U}(1)^V \times \mathrm{U}(1)^A)/\mathbb{Z}_2$ symmetry

$$\text{vector } \mathrm{U}(1)^V\text{:} \quad \Psi_\mathrm{L}^\dagger \mapsto \mathrm{e}^{+\mathrm{i}\theta}\Psi_\mathrm{L}^\dagger \qquad \Psi_\mathrm{R}^\dagger \mapsto \mathrm{e}^{+\mathrm{i}\theta}\Psi_\mathrm{R}^\dagger$$

$$\text{axial } \mathrm{U}(1)^A\text{:} \quad \Psi_\mathrm{L}^\dagger \mapsto \mathrm{e}^{+\mathrm{i}\alpha}\Psi_\mathrm{L}^\dagger \qquad \Psi_\mathrm{R}^\dagger \mapsto \mathrm{e}^{-\mathrm{i}\alpha}\Psi_\mathrm{R}^\dagger$$

➤ Axial charge $Q^A = C_R Q^V C_R^\dagger$, where $C_R\colon \Psi_\mathrm{R} \mapsto \Psi_\mathrm{R}^\dagger$

The chiral anomaly

$$\mathscr{L} = \mathrm{i}\,\Psi_{\mathrm{L}}^{\dagger}(\partial_t + \partial_x)\Psi_{\mathrm{L}} + \mathrm{i}\,\Psi_{\mathrm{R}}^{\dagger}(\partial_t - \partial_x)\Psi_{\mathrm{R}}$$

The chiral anomaly is an anomaly of $(\mathrm{U}(1)^V \times \mathrm{U}(1)^A)/\mathbb{Z}_2$

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Manifests through anomalous current conservation

$$\partial^\mu J_\mu^A = 0 \xrightarrow{\text{turn on } A_\mu} \partial^\mu J_\mu^A = \frac{1}{\pi} E$$

- The obstruction to a trivial gapped phase follows from

  1. Formally: 't Hooft's anomaly matching argument
  2. Physically: threading 2π flux creates $Q^A = 2$ charge—a left-moving particle and right-moving hole

The chiral anomaly

Can the chiral anomaly be realized in a lattice model with finite-dimensional* local Hilbert spaces?

** Bosonized version has been realized in infinite dim local Hilbert spaces [M. Cheng, N. Seiberg '22]*

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$$\left[J_t^V(t,x)\,,J_t^A(t,x')\right] \sim \mathrm{i}\,\partial_x \delta\left(x-x'\right)$$

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More general question: Can the $(\mathrm{U}(1)^V \times \mathrm{U}(1)^A)/\mathbb{Z}_2$ symmetry and its chiral anomaly emanate from a lattice model?

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Yes! [Arkya Chatterjee, **Sal Pace**, Shu-Heng Shao, PRL '25 (arXiv:2409.12220)]

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A simple lattice model

Complex fermions c_j on sites j of length L 1d spatial lattice*

$$\mathscr{H} = \bigotimes_{j=1}^{L} \mathbb{C}^2 \qquad \{c_j, c_{j'}^\dagger\} = \delta_{j,j'} \qquad \{c_j, c_{j'}\} = 0$$

$$H = \mathrm{i} \sum_{j=1}^{L} \left(c_j^\dagger c_{j+1} - c_{j+1}^\dagger c_j \right)$$

Becomes free, massless Dirac fermion theory in IR

* *Assume L is even and periodic boundary conditions*

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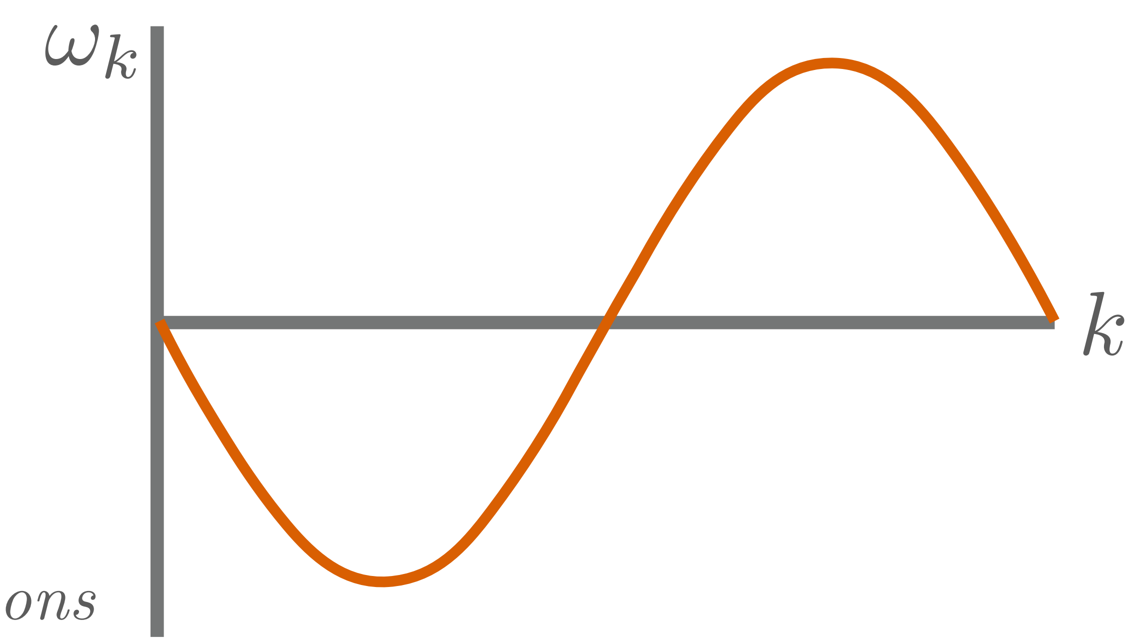
- In momentum space

$$H = \sum_{k \in \mathrm{BZ}} \omega_k c_k^\dagger c_k$$

$$\omega_k = -2\sin(k)$$



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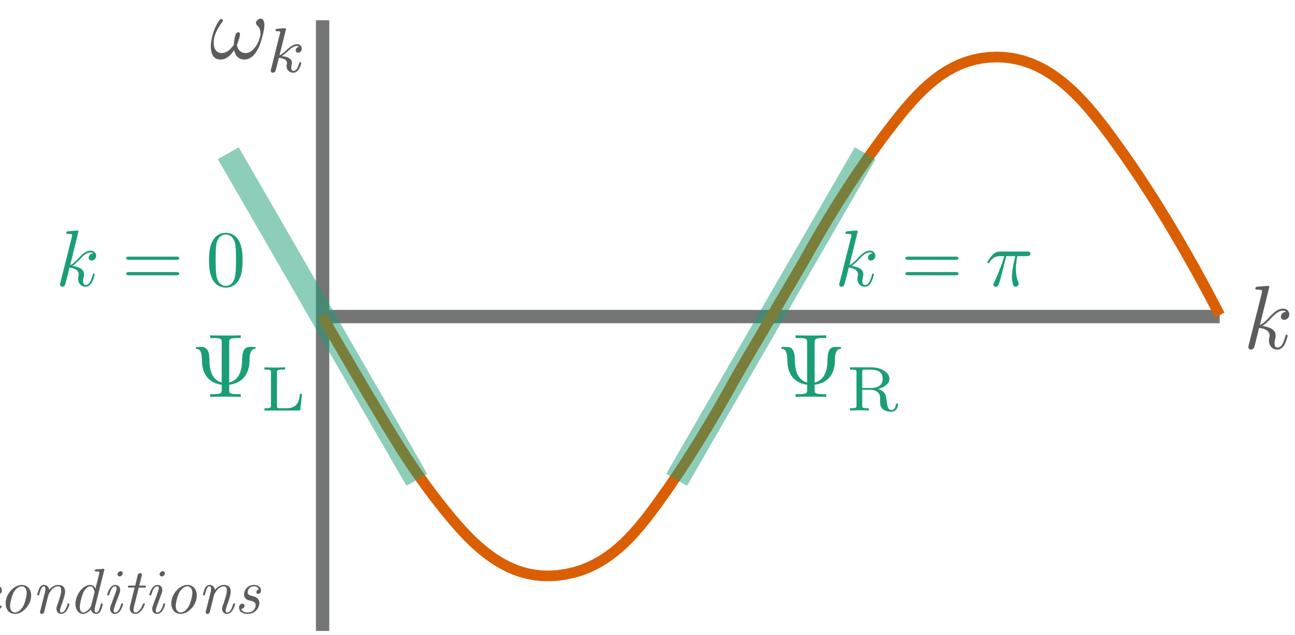

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➤ I

If the chiral anomaly emanates from a lattice anomaly, the chiral $(\mathrm{U}(1)^V \times \mathrm{U}(1)^A)/\mathbb{Z}_2$ symmetry must emanate from a lattice symmetry.

We need to build a UV to IR symmetry dictionary!

*Assume L is even

Emanant symmetries I

$$H = \mathrm{i}\sum_{j=1}^{L}\left(c_j^\dagger c_{j+1} - c_{j+1}^\dagger c_j\right) \qquad c_k = \frac{1}{\sqrt{L}}\sum_{j=1}^{L} \mathrm{e}^{\mathrm{i}kj} c_j$$

U(1) fermion number symmetry $N = \sum_{j=1}^{L} c_j^\dagger c_j$

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U(1) fermion number symmetry $Q_0 = \sum_{j=1}^{L}\left(c_j^\dagger c_j - \frac{1}{2}\right)$

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U(1) fermion number symmetry $Q_0 = \sum_{j=1}^{L} \left(c_j^\dagger c_j - \frac{1}{2} \right)$

- Real space transformation

$$\mathrm{e}^{\mathrm{i}\theta Q_0} : c_j^\dagger \mapsto \mathrm{e}^{\mathrm{i}\theta} c_j^\dagger$$

- Momentum space transformation

$$\mathrm{e}^{\mathrm{i}\theta Q_0} : c_k^\dagger \mapsto \mathrm{e}^{\mathrm{i}\theta} c_k^\dagger$$

- IR symmetry (look at $k = 0$ and $k = \pi$)

$$\mathrm{e}^{\mathrm{i}\theta Q_0} : \Psi_{L,R}^\dagger \mapsto \mathrm{e}^{\mathrm{i}\theta} \Psi_{L,R}^\dagger$$

Emanant symmetries I

Lattice symmetry	$\xrightarrow{\text{emanates to}}$	IR symmetry
$e^{i\theta Q_0}$		$e^{i\theta Q^V}$

$$e^{i\theta Q_0} : \Psi^\dagger_{L,R} \mapsto e^{i\theta} \Psi^\dagger_{L,R}$$

Emanant symmetries II

$$H = \mathrm{i} \sum_{j=1}^{L} \left(c_j^\dagger c_{j+1} - c_{j+1}^\dagger c_j \right) \qquad c_k = \frac{1}{\sqrt{L}} \sum_{j=1}^{L} \mathrm{e}^{\mathrm{i}kj} c_j$$

Lattice translation symmetry:

- Real space transformation

$$T\colon c_j \mapsto c_{j+1}$$

- Momentum space transformation

$$T\colon c_k \mapsto \mathrm{e}^{-\mathrm{i}k} c_k$$

- IR symmetry (look at $k = 0$ and $k = \pi$)

$$T\colon \Psi_{\mathrm{L}}, \Psi_{\mathrm{R}} \mapsto \Psi_{\mathrm{L}}, -\Psi_{\mathrm{R}}$$

Emanant symmetries II

Lattice symmetry	$\xrightarrow{\text{emanates to}}$	IR symmetry
$e^{i\theta Q_0}$		$e^{i\theta Q^V}$
T		$e^{i\pi Q_R} = e^{\frac{i\pi}{2}(Q^V - Q^A)}$

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T	$\mathrm{e}^{\mathrm{i}\pi Q_R} = \mathrm{e}^{\frac{\mathrm{i}\pi}{2}(Q^V - Q^A)}$

$\mathrm{e}^{\mathrm{i}\pi Q_R}$ generates an anomalous chiral symmetry in the IR

- $\mathbb{Z}_2^R$ emanates from lattice translations
- But T is anomaly-free: this IR anomaly is emergent

$$T\colon \Psi_{\mathrm{L}}, \Psi_{\mathrm{R}} \mapsto \Psi_{\mathrm{L}}, -\Psi_{\mathrm{R}}$$

Be real 😎

Let's decompose c_j into real (Majorana) fermions $a_j = a_j^\dagger$ and $b_j = b_j^\dagger$ to search for other useful symmetries

$$c_j = \frac{1}{2}(a_j + \mathrm{i} b_j) \qquad \{a_j, a_{j'}\} = 2\delta_{j,j'} \qquad \{b_j, b_{j'}\} = 2\delta_{j,j'}$$

Hamiltonian becomes $H = \dfrac{\mathrm{i}}{2}\displaystyle\sum_{j=1}^{L}(a_j a_{j+1} + b_j b_{j+1})$

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- The a_j and b_j Majoranas are decoupled!
- The model has Majorana translation symmetries

$$T_a : a_j, b_j \mapsto a_{j+1}, b_j$$

$$T_b : a_j, b_j \mapsto a_j, b_{j+1}$$

Emanant symmetries III

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The b Majorana lattice translation symmetry

- Real space transformation:

$$T_b : c_j \mapsto \frac{1}{2}\left(a_j + \mathrm{i}b_{j+1}\right)$$

- Momentum space transformation

$$T_b : c_k = \frac{1}{2}\left(a_k + \mathrm{i}b_k\right) \mapsto \frac{1}{2}\left(a_k + \mathrm{e}^{-\mathrm{i}k}\mathrm{i}b_k\right)$$

- IR symmetry (look at $k=0$ and $k=\pi$)

$$T_b : \Psi_{\mathrm{L}}, \Psi_{\mathrm{R}} \mapsto \Psi_{\mathrm{L}}, \Psi_{\mathrm{R}}^{\dagger}$$

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T_b		C_R

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Emanant symmetries III

Lattice symmetry $\xrightarrow{\text{emanates to}}$	IR symmetry
$\mathrm{e}^{\mathrm{i}\theta Q_0}$	$\mathrm{e}^{\mathrm{i}\theta Q^V}$
T	$\mathrm{e}^{\mathrm{i}\pi Q_R} = \mathrm{e}^{\frac{\mathrm{i}\pi}{2}(Q^V - Q^A)}$
T_b	C_R
$\mathrm{e}^{\mathrm{i}\alpha\,(T_b Q_0 T_b^\dagger)}$	$\mathrm{e}^{\mathrm{i}\alpha\,(C_R Q^V C_R^\dagger)} = \mathrm{e}^{\mathrm{i}\alpha Q^A}$

$$T_b : \Psi_{\mathrm{L}}, \Psi_{\mathrm{R}} \mapsto \Psi_{\mathrm{L}}, \Psi_{\mathrm{R}}^\dagger$$

Lattice vector and axial charges

The IR vector and axial charges emanate from the conserved charges

- Lattice vector charge $Q_0 = \frac{\mathrm{i}}{2} \sum_{j=1}^{L} a_j b_j$

- Lattice axial charge $Q_1 \equiv T_b Q_0 T_b^{-1} = \frac{\mathrm{i}}{2} \sum_{j=1}^{L} a_j b_{j+1}$

  1. Sum of local terms
  2. Have integer-quantized eigenvalues
  3. Generate locality preserving U(1) symmetries

Onsager symmetry

The lattice vector and axial charges do not commute

$$[Q_0, Q_1] \neq 0 \xrightarrow{\text{IR limit}} [Q^V, Q^A] = 0$$

Q_0 and Q_1 generate the Onsager algebra [Onsager '44]

- Let $Q_n = \frac{\mathrm{i}}{2}\sum_{j=1}^{L} a_j b_{j+n}$ and $G_n = \frac{\mathrm{i}}{2}\sum_{j=1}^{L}(a_j a_{j+n} - b_j b_{j+n})$

$$[Q_n, Q_m] = \mathrm{i} G_{m-n} \qquad [G_n, G_m] = 0$$

$$[Q_n, G_m] = 2\mathrm{i}(Q_{n-m} - Q_{n+m})$$

Onsager symmetry

The lattice vector and axial charges do not commute

The chiral $(\mathrm{U}(1)^V \times \mathrm{U}(1)^A)/\mathbb{Z}_2$ symmetry emanates from the Onsager symmetry $\langle \mathrm{e}^{\mathrm{i}\theta Q_0}, \mathrm{e}^{\mathrm{i}\alpha Q_1} \rangle$

$$Q_n \xrightarrow{\text{IR limit}} \begin{cases} Q^V & n \text{ even} \\ Q^A & n \text{ odd} \end{cases} \qquad\qquad G_n \xrightarrow{\text{IR limit}} 0$$

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Does the Onsager symmetry have a lattice anomaly that matches the chiral anomaly?

Onsager symmetric Hamiltonians

We assume the Hamiltonian is local:

$$H_g = \sum_n \sum_{j=1}^{L} g_{j,n} H_j^{(n)}$$

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1. $e^{-\mathrm{i}\frac{\pi}{2}Q_1} e^{\mathrm{i}\frac{\pi}{2}Q_0} : (a_j, b_j) \mapsto (a_{j-1}, b_{j+1})$ invariance requires $H_j^{(n)}$ to <u>not</u> have terms mixing a_j and b_j and $g_{j,n} = g_n$

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2. Under the $\mathrm{e}^{\mathrm{i}\theta Q_0}$ transformation

$$a_j \to \cos(\theta) a_j + \sin(\theta) b_j \qquad b_j \to \cos(\theta) b_j - \sin(\theta) a_j$$

$\implies$ Symmetric $H_j^{(n)}$ are quadratic

$$H_j^{(n)} = \mathrm{i} a_j a_{j+n} + \mathrm{i} b_j b_{j+n}$$

Lattice anomaly: enforced gaplessness

$$H_g = \mathrm{i} \sum_n \sum_{j=1}^{L} g_n \left(a_j a_{j+n} + b_j b_{j+n} \right)$$

- H_g commutes with the entire Onsager symmetry — it is the most general Onsager symmetric Hamiltonian

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Every Onsager symmetric Hamiltonian H_g is gapless

- In momentum space:

$$H_g = \sum_{k\in\mathrm{BZ}} \omega_k c_k^\dagger c_k \qquad \omega_k = 4\sum_n g_n \sin(nk)$$

- H_g is never in a trivial gapped phase
- This Onsager symmetry has a lattice anomaly

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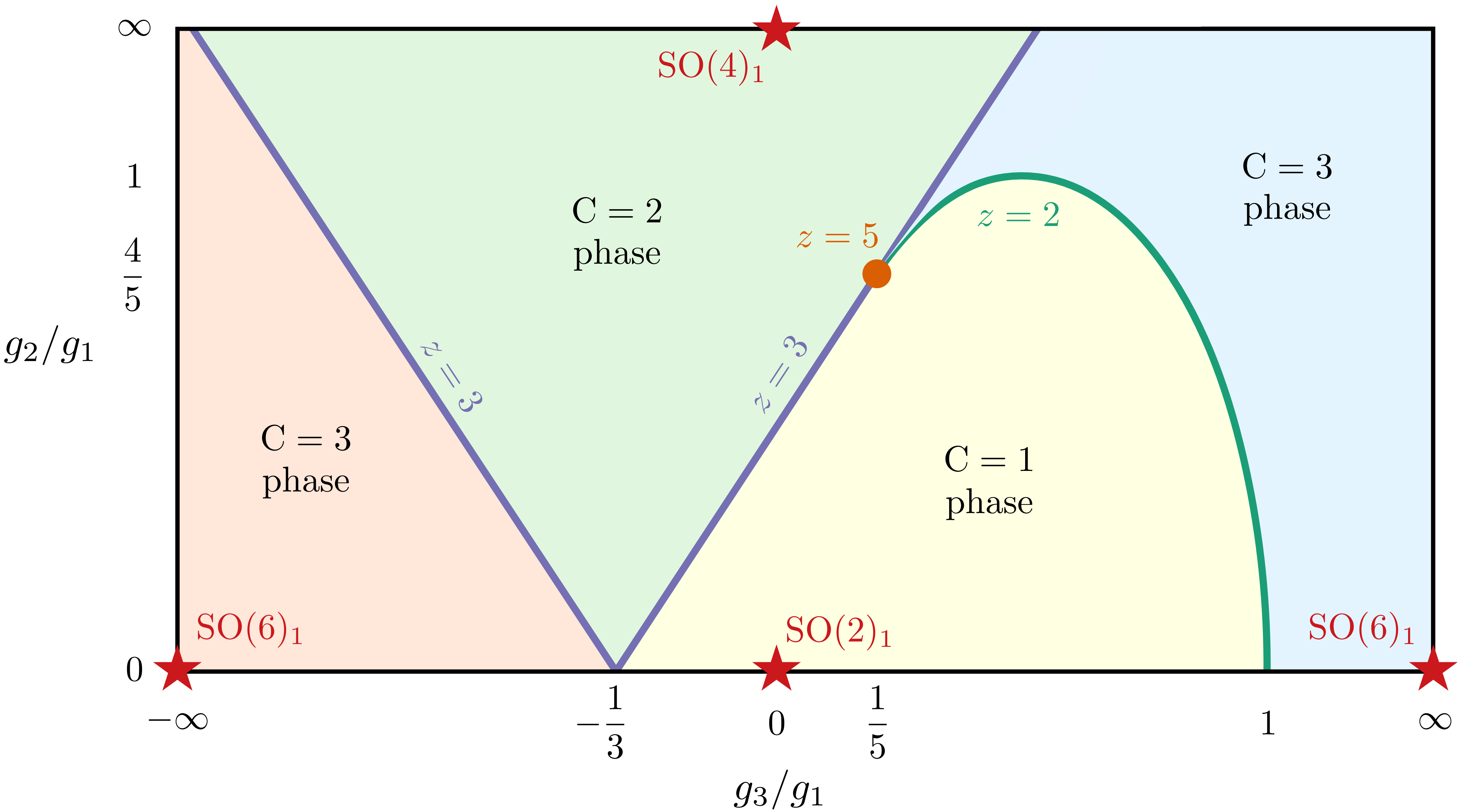
The chiral anomaly emanates from the Onsager symmetry's lattice anomaly

- The fact that the lattice anomaly enforces gaplessness is consistent—the chiral anomaly is a local anomaly and enforces gaplessness in QFT

- H_g is never in a trivial gapped phase
- This Onsager symmetry has a lattice anomaly

Gapless phase diagram

$$H = \mathrm{i}\sum_{n=1}^{3}\sum_{j=1}^{L} g_n\left(a_j a_{j+n} + b_j b_{j+n}\right)$$



Summary

Anomalies 101

1. Anomalies as obstructions to trivial gapped phases
2. Emergent vs emanant anomalies

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Lattice chiral anomaly [Arkya Chatterjee, **Sal Pace**, Shu-Heng Shao, PRL '25 (arXiv:2409.12220)]

The simple tight-binding model $H = \mathrm{i}\sum_{j=1}^{L}\left(c_j^\dagger c_{j+1} - c_{j+1}^\dagger c_j\right)$

1. Has a lattice vector and axial symmetry, which form the Onsager algebra
2. Has a lattice anomaly that enforces gaplessness, from which the chiral anomaly emanates

Summary

Anomalies 101

Other lattice anomalies from Onsager-like symmetries:

- Compact boson CFT anomalies in spin chains
  [**Sal Pace**, Arkya Chatterjee, Shu-Heng Shao, SciPost Phys '25 (arXiv:2412.18606)]
- Witten's SU(2) anomaly on the lattice
  [L. Gioia, R. Thorngren, arXiv:2503.07708]
- Lattice parity anomaly: symmetry-enforced Dirac cones
  [**Sal Pace**, Luke Kim, Arkya Chatterjee, Shu-Heng Shao, arXiv:2505.04684]
- Lattice LU(1) anomaly: symmetry-enforced Fermi surfaces
  [Luke Kim, **Sal Pace**, Shu-Heng Shao, *to appear*]

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Back-up slides

Lattice vector and axial charges II

In terms of complex fermions

- Lattice vector charge $Q_0 = \sum_{j=1}^{L} \left(c_j^\dagger c_j - \frac{1}{2} \right)$
- Lattice axial charge

$$Q_1 = \frac{1}{2} \sum_{j=1}^{L} \left(c_j^\dagger c_{j+1} - c_j c_{j+1}^\dagger + c_j c_{j+1} - c_j^\dagger c_{j+1}^\dagger \right)$$

- Generate locality preserving U(1) symmetries

$$\mathrm{e}^{\mathrm{i}\theta Q_0} : c_j \mapsto \mathrm{e}^{-\mathrm{i}\theta} c_j$$

$$\mathrm{e}^{\mathrm{i}\alpha Q_1} : c_j \mapsto \cos(\alpha) c_j - \frac{\mathrm{i}}{2} \sin(\alpha) (c_{j-1}^\dagger + c_{j-1} - c_{j+1}^\dagger + c_{j+1})$$

An anomaly in quantum spin chains

Consider model with a qubit on each site j of a length L ring

$$\mathscr{H} = \bigotimes_{j=1}^{L} \mathbb{C}^2 \qquad X_j = X_{j+L} \qquad Z_j = Z_{j+L}$$

Anomalous $\mathbb{Z}_2^X \times \mathbb{Z}_2^Z \times$(lattice translations) symmetry

$$U_X = \prod_{j=1}^{L} X_j \qquad U_Z = \prod_{j=1}^{L} Z_j \qquad T : j \mapsto j+1$$

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Manifestations of the anomaly:

- U_X and U_Z have a QM anomaly in each unit cell
- $U_X U_Z = (-1)^L U_Z U_X$

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Every symmetric Hamiltonian cannot have a trivial gapped phase

- e.g., XX chain $H = \sum_{j=1}^{L} (X_j X_{j+1} + Y_j Y_{j+1})$
- Called a Lieb-Schultz-Mattis (LSM) anomaly

An anomaly in field theory

The compact boson CFT at radius R is a $1+1$D CFT with

$$\mathscr{L}_R = \frac{R^2}{4\pi}\partial_\mu\Phi\partial^\mu\Phi \qquad\qquad \Phi \sim \Phi + 2\pi$$

- Has a $(\mathrm{U}(1)^M \times \mathrm{U}(1)^W) \rtimes \mathbb{Z}_2^C$ symmetry:

$$J^M_\mu = \frac{R^2}{2\pi}\partial_\mu\Phi \qquad\qquad J^W_\mu = \frac{1}{2\pi}\epsilon_{\mu\nu}\partial^\nu\Phi \qquad\qquad C : \Phi \mapsto -\Phi$$

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Anomalous symmetries:

- $\mathrm{U}(1)^M \times \mathrm{U}(1)^W$ and $\mathbb{Z}_2^M \times \mathbb{Z}_2^W \times \mathbb{Z}_2^C$
- Manifestation of $\mathrm{U}(1)^M \times \mathrm{U}(1)^W$ anomaly:

$$\partial^\mu J^W_\mu = 0 \xrightarrow{\text{turn on } A^M_\mu} \partial^\mu J^W_\mu = \frac{1}{2\pi}E^M$$

Anomaly matching

Anomalies of the effective IR theory either emerge from nothing or emanate from an anomaly of the UV

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Example: XX spin chain $\longrightarrow$ $R = \sqrt{2}$ boson CFT

$\mathbb{Z}_2^X$	$\mathbb{Z}_2^C$
$\mathbb{Z}_2^Z$	$\mathbb{Z}_2^M$
Translations	$\mathbb{Z}_2^{\text{diag}} \subset \mathbb{Z}_2^M \times \mathbb{Z}_2^W$

[M. A. Metlitski, Thorngren '17; M. Cheng, N. Seiberg '22]

- Anomaly of $\mathbb{Z}_2^{\text{diag}}$ is emergent
- Anomaly of $\mathbb{Z}_2^M \times \mathbb{Z}_2^W \times \mathbb{Z}_2^C$ emanates from LSM anomaly