# A Classification of defect-free disordered Phases

**Based on:**

SP arXiv: 2308.05730

SP, C. Zhu, A. Beaudry, X-G Wen arXiv: 2310.08554

**Outline:**

I) ordered and (defect-free) disordered phases

II) Generalized Sym. in ordered phases

III) Classification of Defect-free Disordered phases

**Ordered phases:**

- Internal $G$ Sym. is spontaneously broken:

$$G \xrightarrow{SSB} H \subset G$$

- Order parameter $\quad u(x) \in G/H \equiv \{gH : g \in G\}$

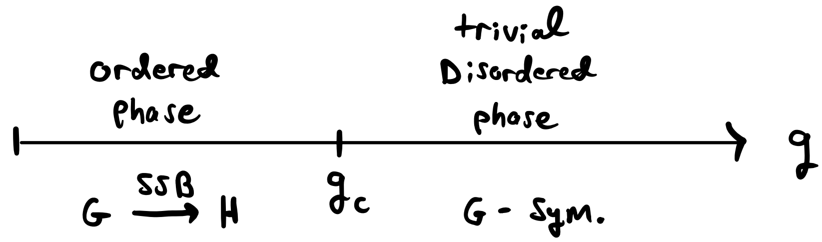
→ Ising Ferromagnet : $\mathbb{Z}_2 \xrightarrow{SSB} \mathbb{1}$ , $G/H \simeq \mathbb{Z}_2$

Superfluid: $U(1) \xrightarrow{SSB} \mathbb{1}$ , $G/H \simeq S^1$

AFM : $SU(2) \xrightarrow{SSB} U(1)$ , $G/H \simeq S^2$

**Transitions out of ordered phases:**

**Standard Story: Landau-Ginzburg Paradigm**




→ Sym restored by condensing topological defects (eg. Domain walls, Hedgehogs, Dislocations, etc)

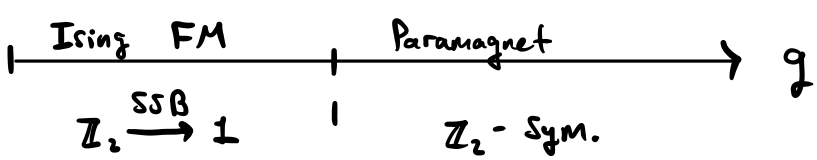
Ex) 1d transverse field Ising model

→ Qubits on sites $j$ of infinite chain

$$H = -\sum_j Z_j Z_{j+1} - g \sum_j X_j$$

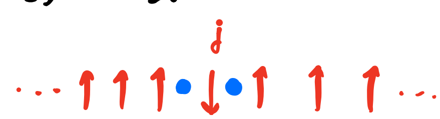
→ $\mathbb{Z}_2$ symmetry generated by $U = \prod_j X_j$

→ Phase diagram




→ $\mathbb{Z}_2$ domain walls:
⇒ classified by $\pi_0(G/H \simeq \mathbb{Z}_2) \simeq \mathbb{Z}_2$.

1) excited by $X_j$




2) gapped excitations in ordered phase
3) Condensed in disordered phase
⇒ $\langle \prod_{i<j<k} X_j \rangle \sim \mathcal{O}(1)$ for $|i-k| \gg 1$.

Can label phases as

| ordered phase | $g_c$ | Disordered phase | g |
|---|---|---|---|
| G Sym charges condensed, top. defects gapped | | G Sym charges gapped, top. defects condensed | |

## what if topological defects are static?

More gen. just highly suppresed,

- → Do not move under time evolution (eg, Ising model with $g=0$)
- → top. defects can no longer condense:

  Q: Can disordered phases be realized?

  A: Yes!

Example: 2+1 D $O(3)$ Model [Kamal, Murthy '93 / Motrunich, Vishwanath '04]

$$Z = \int D\vec{n} \, \exp\left[i \frac{1}{g} \int (\partial \vec{n})^2\right] \quad \text{with} \quad \vec{n} \in S^2$$

→ top. defects: $\pi_2(S^2) \simeq \mathbb{Z} \Rightarrow$ Skyrmions

$$J^\mu = \frac{1}{8\pi} \epsilon^{\mu\nu\rho} \, \vec{n} \cdot (\partial_\nu \vec{n} \times \partial_\rho \vec{n})$$

Standard Story

1) Regularize path integral while allowing dynamical Skyrmions

Ordered | Disordered

$g=0$ — $\langle \vec{n} \rangle \neq 0$ — $g_c$ — $\langle \vec{n} \rangle = 0$ & exponentially decaying correlations — $g=\infty$

2) Regularize path integral without dynamical Skyrmions

→ $\partial_\mu J^\mu = 0$ constraint.

→ Convinient to use $\mathbb{CP}^1$ presentation of $O(3)$ model

$$\vec{n} = z^\dagger \vec{\sigma} z$$ } z transforms under spinor rep of $SO(3)$

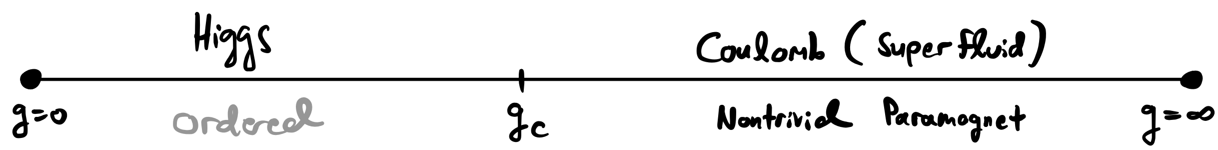
with $z \in \mathbb{C}^2$ and $z^\dagger z = 1$, $\vec{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$, $U(1)$ gauge field

$$A_\mu = i\, z^\dagger \partial_\mu z \quad \Rightarrow \quad J^\mu = \varepsilon^{\mu\nu\rho} \partial_\nu A_\rho$$

→ $\partial_\mu J^\mu = 0$ means no U(1) Magnetic instantons

Static Skyrmions

→ Effective theory : *non-compact* U(1) gauge theory w/ matter $z$




$\langle \vec{n} \rangle \neq 0$ $\qquad$ $\langle \vec{n} \rangle = 0$ & algebraic decaying correlations

**A Symmetry perspective on the nontrivial disordered phase ?**

Static topological defects

↓

their topological charge is conserved

↓

New global symmetry : $S_\pi$

2+1D O(3) revisited :

→ Only static Skyrmions ⇒ $\partial_\mu J^\mu = 0$

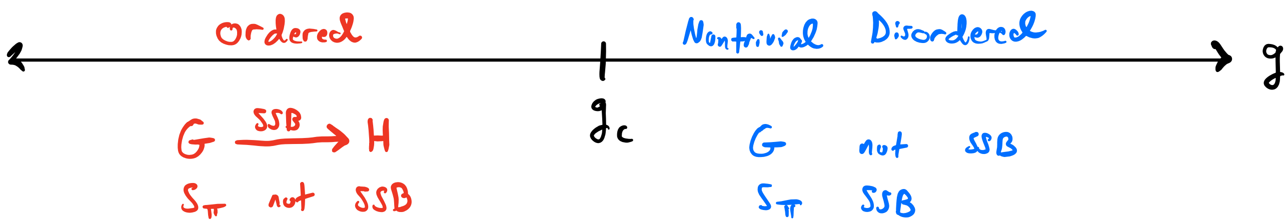
$$N = \int d^2\vec{x}\; J^0 \in \mathbb{Z} \quad \text{is conserved}$$

→ $S_\pi = U(1)$ and Sym. operator is $e^{i\alpha N}$.

⇒ So $S_\pi$ SSB phase is Skyrmion SF.

→ SSB patterns of $S_\pi$ classify the nontrivial disordered phases.

**Goal:** Classify "defect-free" disordered phases using $S_\pi$




- Top. defects are very rich

→ Extended objects (eg. O(3) model in higher dimensions)

→ Defects of different dimensions can have nontrivial interplay (eg. Nematic liquid crystals)

→ Defects fusion rules exotic (eg. non-abelian Vortices)

Need <u>generalized symmetries</u>

## Generalized Symmetries

Ordinary Symmetries in QM:

1) described by a group G
2) represented by (anti-)Unitary operators $U_g$ $(g \in G)$

$\Rightarrow U_g \times U_h = U_{gh}$

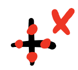
3) act uniformly on <u>all</u> degrees of freedom in Space

Two generalizations: (There are More!)

1) n-Form Symmetry (ordinary Symmetry: $n=0$)

$\Rightarrow$ Instead of acting on all of d-dim space, acts on a closed $(d-n)$-dim manifold in space.

$\Rightarrow$ Conserved quantities are n-dim objects eg. 2+1D toric code

$H = -\sum_s$  $-\sum_p$ 

Commutes with $U(\gamma) = \prod_{e \in \gamma} Z_e$ } $\mathbb{Z}_2$ 1-form Symmetry

$\rightarrow U(\gamma)\, W(\hat{\gamma})\, U^\dagger(\gamma) = (-1)^{\mathrm{int}(\gamma, \hat{\gamma})}\, W(\hat{\gamma})$

$\rightarrow$ $\mathbb{Z}_2$ topological order arises from this Symmetry spontaneously breaking.

2) Non-invertible Symmetry

$\Rightarrow$ Symmetry operators have a non-trivial Kernel

$\Rightarrow$ No longer described by groups, obey

$$S_a \times S_b = \sum_c N_{ab}^c\, S_c$$

eg 1+1D critical Ising model $H = -\sum_j Z_j Z_{j+1} + X_j$

Commutes with operator $D$ obeying

$$D\, Z_j Z_{j+1} = X_{j+1}\, D \qquad D\, X_j = Z_j Z_{j+1}\, D$$

lattice translations.

$$D \times D = T + T \prod_j X_j$$

- Generalized Symmetries can

1) Constrain correlation functions

2) Spontaneously break

3) Emerge at long-distances / low energy and at critical points

4) Characterize SPT phases

Passes the Duck test!

**What is $S_\pi$ ?**

**Consider O(3) model now in 3+1 D.**

→ $\pi_2(S^2) \simeq \mathbb{Z}$ defects are Hedgehogs ⟹ Strings in Spacetime

→ Hedgehog String current $J^{\mu\nu} = \frac{1}{8\pi} \varepsilon^{\mu\nu\rho\sigma} \vec{n} \cdot (\partial_\rho \vec{n} \times \partial_\sigma \vec{n})$

→ No dynamical Hedgehogs ⟹ $\partial_\mu J^{\mu\nu} = 0$

$$N(\Sigma) = \int_\Sigma J^{0i} \hat{n}^i \, dS \in \mathbb{Z} \quad \text{is conserved}$$

Surface

→ Symmetry defect $e^{i\alpha N(\Sigma)}$

1) detects Hedgehog Strings
2) generates a $U(1)$ $3-2=1$ form Sym.

$$\boxed{S_\pi = U(1)^{(1)}}$$

→ Phase diagram follows from $\mathbb{CP}^1$ presentation
(Hedgehogs = Magnetic Monopoles)


$\langle \vec{n} \rangle \neq 0$
- $SO(3) \to SO(2)$
- $U(1)^{(1)}$ not SSB

Monopoles confined

$\langle \vec{n} \rangle = 0$
- $SO(3)$ not SSB
- $U(1)^{(1)}$ SSB.

Monopoles deconfined

→ Two lessons:

1) SSBing higher form symmetry ⇒ deconfinement.
2) Nontrivial paramagnetic phases classified by

$$U(1)^{(1)} \xrightarrow{SSB} \mathbb{Z}_N^{(1)} \quad , \quad \mathbb{Z}_N^{(1)} \quad \text{SPT}$$

→ labeled by $n \in \mathbb{Z}$ and $m \in \mathbb{Z}_n$.

Consider $SO(3) \xrightarrow{SSB} \mathbb{Z}_2 \times \mathbb{Z}_2$ in 2+1 D.

→ $\pi_1(SO(3)/\mathbb{Z}_2 \times \mathbb{Z}_2) \simeq Q_8$ defects are non-abelian vortices, classified by conjugacy classes of $Q_8$.

⇒ $Q_8$ is quaternion group $(\pm \mathbb{1}, \pm i\sigma^{x,y,z})$

⇒ $Cl(Q_8) = \{[1], [-1], [\pm i\sigma^x], [\pm i\sigma^y], [\pm i\sigma^z]\}$

→ No dynamical vortices ⇒ conserved $Cl(Q_8)$ strings

What is $S_\pi$??
⇒ Spoilers:

1) $S_\pi$ is a non-invertible 1-form sym.
2) $S_\pi$ SSB phases are abelian and non-abelian SETs.

→ Let's show this within a toy 2+1 D Hamiltonian model
↳ schematic!

1) degrees of freedom on square lattice:

- SU(2) rotors on sites
  → Transform under the SO(3) symmetry
- $Q_8$ gauge fields on links.

⇒ Why?

- $Q_8$ gauge redundancy compactifies

$$SU(2) \to SU(2)/Q_8 \simeq SO(3)/\mathbb{Z}_2 \times \mathbb{Z}_2$$

$\rightarrow$ SU(2) rotors is $SO(3) \xrightarrow{SSB} \mathbb{Z}_2 \times \mathbb{Z}_2$ order parameter

- $\pi_1(SO(3)/\mathbb{Z}_2 \times \mathbb{Z}_2) \simeq \pi_0(Q_8)$

$\rightarrow$ non-abelian vortices are $Q_8$ magnetic fluxes

2) Hamiltonian (Schematically)

$$H = H_{Q_8} + \lambda H_{Higgs}$$

$Q_8$ pure gauge theory (ie, quantum double model)

Minimal coupling SU(2) rotors w/ $Q_8$ gauge field (ie String operators)

$\rightarrow$ Symmetry of H: $W_R(\gamma)$

$\Rightarrow$ $\gamma$ is closed path

$\Rightarrow$ R is irrep of $Q_8$

trivial : $\mathbb{1}$

Signs : $P_1$, $P_2$, $P_3$

2d : $E$

$Q_8$ Wilson loop operator

$\rightarrow$ Detects Magnetic fluxes!

$\Rightarrow$ obeys : $W_{P_j}(\gamma) \times W_{P_j}(\gamma) = 1$

$$W_E(\gamma) \times W_E(\gamma) = 1 + \sum_j W_{P_j}(\gamma)$$

$\Rightarrow$ Non-invertible : $\boxed{\text{Rep}(Q_8) \text{ 1-form Symmetry}}$

$\rightarrow$ Phase diagram

Non-abelian Vortices deconfine.

Higgs = ordered — $Q_8$ top. order. = Nontrivial Paramagnet

$0$ — $SO(3) \to \mathbb{Z}_2 \times \mathbb{Z}_2$ — $1/\lambda_c$ — $\mathrm{Rep}(Q_8)^{(1)}$ SSB — $\infty$

→ classification: $\mathrm{Rep}(Q_8)$ has five SSB patterns

⇒ five nontrivial paramagnet phases.

## $S_\pi$ and defect-free disordering

- Each nontrivial $\pi_K(G/H) \Rightarrow (d-K)$-form Symmetry.
- $S_\pi$ is not SSB'd in the ordered phase

  → why? Top defects are confined

- Spontaneously breaking $S_\pi$ drives a transition out of the ordered phase.

  → causes top defects to deconfine

Related to an 't Hooft anomaly

  → Since ordered ground states confine top defects, SSBing $S_\pi$ should restore $G$.

## General aspects in 2+1 D

(Skip if short on time)

For a $G \xrightarrow{SSB} H$ Phase w $G/H$ path connected top defects are classified by

1) $\pi_1(G/H)$ : Vortex Strings
2) $\pi_2(G/H)$ : Instanton events

3) Action of $\pi_1$ on $\pi_2$

→ Braiding $\pi_2$ defect around $\pi_1$ can change topological charge of $\pi_2$

4) 3-cocycle : $\pi_1 \times \pi_1 \times \pi_1 \to \pi_2$

→ $\pi_1$ defects can be deformed into a $\pi_2$ defect

⇒ $\pi_1$ defects carry $\pi_2$ topological charge.

⇝ Data packaged into Math object $\mathbb{G}^{(2)}$.

⇝ Can show that with all top. defects static

$$S_\pi = 2\text{-Rep}(\mathbb{G}^{(2)})$$

⇒ called a Monoidal 2-category

⇝ Nontrivial disordered phases classified by

1) $S_\pi$ SSB patterns

2) Residual Sym. SPTs.

## Summary

1) generalized Sym. appear in defect-free ordered phases (Can emerge in generic ordered phases)

2) They can be used to classify + predict exotic neighboring disordered phases

Only Scratched the Surface!